Exact half-BPS type IIB interface solutions I: local solution and supersymmetric Janus

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ABSTRACT: The complete Type IIB supergravity solutions with 16 supersymmetries are obtained on the manifold $AdS_4 \times S^2 \times S^2 \times \Sigma$ with $SO(2,3) \times SO(3) \times SO(3)$ symmetry in terms of two holomorphic functions on a Riemann surface Σ, which generally has a boundary. This is achieved by reducing the BPS equations using the above symmetry requirements, proving that all solutions of the BPS equations solve the full Type IIB supergravity field equations, mapping the BPS equations onto a new integrable system akin to the Liouville and Sine-Gordon theories, and mapping this integrable system to a linear equation which can be solved exactly. Amongst the infinite class of solutions, a non-singular Janus solution is identified which provides the AdS/CFT dual of the maximally supersymmetric Yang-Mills interface theory discovered recently. The construction of general classes of globally non-singular solutions, including fully back-reacted $AdS_5 \times S^5$ and supersymmetric Janus doped with D5 and/or NS5 branes, is deferred to a companion paper [1].

Contents

1. Introduction

A particularly interesting application of the AdS/CFT correspondence [2 – 4] (for reviews, see [5, 6]) is provided by conformal field theory (CFT) in the presence of a planar interface or a planar defect.[1] The addition of a planar interface to four-dimensional $\mathcal{N} = 4$ super Yang-Mills, (specified by interface couplings of local bulk operators which are supported only on the interface) already gives rise to a rich family of interface CFTs. In particular, it was shown in [7] that, while the conformal symmetry group SO(2,4) of the $\mathcal{N} = 4$ theory is always reduced to the conformal group SO(2,3) of the planar interface, the 32 conformal supersymmetries of the bulk theory may be reduced to either 0, 4, 8, or 16 conformal supersymmetries, and maximal internal symmetry groups of SO(6), SU(3), SU(2) × U(1) and SO(3) × SO(3) respectively.

The AdS/CFT duals of conformal interface and defect theories reflect the residual conformal group SO(2,3) of the planar interface, and correspond to Type IIB superstring theory (or its Type IIB supergravity limit) on a warped space containing AdS_4, since the isometry group of AdS_4 is precisely SO(2,3). For example, the intersection of D3 and probe D5 branes produces AdS/CFT duals to planar defect theories, where the extra degrees of freedom are produced by the dynamics of open strings spanned between the various intersecting branes [8, 9, 11 – 13, 10].

The original Janus solution of [14] is AdS/CFT dual to the interface Yang-Mills theory with 0 supersymmetries listed at the end of the first paragraph. (see [15 – 19] for other developments on the Janus solution). The Janus solution is a 1-parameter family of dilatonic deformations of $AdS_5 \times S^5$ in which the entire internal symmetry SO(6) is preserved, but supersymmetry is completely broken. Nonetheless, Janus is stable against all small and a certain class of large perturbations [20, 21]. Its geometry is $AdS_4 \times \mathbf{R} \times S^5$, where $\mathbf{R}$ parametrizes the varying dilaton, and is of co-homogeneity 1. The AdS/CFT dual interface theory is pure $\mathcal{N} = 4$ super-Yang-Mills on either side of the interface, across which the

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[1] We distinguish between the interface and defect theories as follows. Compared to the bulk theory, the defect theory has extra degrees of freedom localized on the defect, while the interface does not.

gauge coupling varies discontinuously. Several dynamical problems in the interface Yang-Mills theory, such as the persistence of the interface conformal symmetry at the quantum level, may be addressed by directly exploiting the dynamics of the bulk theory [22, 23].

A 2-parameter family of supersymmetric Janus solutions to Type IIB supergravity was obtained in [24] (see also [25]). With its 4 supersymmetries, and SU(3) internal symmetry, it emerged as a natural AdS/CFT dual to the interface theory with 4 supersymmetries listed at the end of the first paragraph. Its geometry is now $AdS_4 \times \mathbf{R} \times CP_2 \times_1 S^1$, and is of co-homogeneity 1. Here, $CP_2 \times_1 S^1$ is topologically S^5, but isometric only under the SU(3) × U(1) subgroup of the isometry group SU(4) of S^5. This space was encountered earlier in the context of supergravity solutions in [26−28].

The initial motivation for the present work was to obtain a Janus solution of Type IIB supergravity which is dual to the interface Yang-Mills theory with 16 supersymmetries, listed at the end of the first paragraph. The geometry of the solution is in part determined by the conformal SO(2, 3), and the internal SO(3) × SO(3) symmetries of the Yang-Mills interface theory, which require a manifold $AdS_4 \times M_6$ where M_6 has SO(3)×SO(3) isometry and the product is warped over M_6. There are many possible such M_6 spaces. The particular reduction of SO(6) internal symmetry on the six scalars of the Yang-Mills theory, obtained in [7], lead one to conclude that M_6 is a warping of $S^2 \times S^2$, which manifestly exhibits the desired SO(3) × SO(3) isometry.

The initial motivation described above, namely a search for a Janus solution with 16 supersymmetries, thus leads one to consider Type IIB supergravity on the following spaces,

$$AdS_4 \times S^2 \times S^2 \times \Sigma \tag{1.1}$$

with SO(2, 3)×SO(3)×SO(3) isometry. In general, the product spaces are warped over the two-dimensional parameter space Σ, which is a Riemann surface with boundary, and these spaces are of co-homogeneity 2. A further motivation for considering Type IIB solutions on these spaces derives from the similarity of this problem to the one of "bubbling AdS space and 1/2 BPS geometries" of [29] (see also [30]). The Killing spinors and the reduced BPS equations for this case were calculated by Gomis and Römelsberger [31], but the only explicit solution obtained there was $AdS_5 \times S^5$.

In the present paper, we shall derive all Type IIB supergravity solutions with 16 super-symmetries and $AdS_4 \times S^2 \times S^2 \times \Sigma$ space-time geometry with SO(2, 3) × SO(3) × SO(3) symmetry, in terms of two harmonic functions h_1 and h_2 on Σ. In general, these solutions have varying dilaton ϕ and non-vanishing 3-form field strengths. For example, the dilaton field for the general solution takes the following form,

$$e^{4\phi} = \frac{2h_1 h_2 |\partial_w h_2|^2 - h_2^2(\partial_w h_1 \partial_{\bar{w}} h_2 + \partial_w h_2 \partial_{\bar{w}} h_1)}{2h_1 h_2 |\partial_w h_1|^2 - h_1^2(\partial_w h_1 \partial_{\bar{w}} h_2 + \partial_w h_2 \partial_{\bar{w}} h_1)} \tag{1.2}$$

for any local complex coordinate w on Σ. Other fields are given by analogous explicit expressions in terms of h_1 and h_2, which will be derived and presented in section 9.

Some of these solutions are everywhere non-singular, while others have singularities. The analysis in this paper is mostly restricted to the local structure of the solutions and we defer to a companion paper [1] the study of global properties and singularities such as

those of the D5 and NS5 brane type. Amongst the regular solutions, we readily identify in section 10 of this paper one family which is of the Janus type. By construction, this solution has 16 supersymmetries and $SO(3) \times SO(3)$ internal symmetry, as was hoped for.

The complete and exact solution to the reduced BPS equations is constructed by mapping the BPS equations onto a seemingly new integrable system, which is akin to the Liouville and Sine-Gordon theories. Its field equation is given by,

$$\partial_{\bar{w}}\partial_w \vartheta + \frac{1}{\cos\mu}\left(e^{-i\vartheta}\,\partial_{\bar{w}}\vartheta\,\partial_w\lambda + e^{i\vartheta}\,\partial_w\vartheta\,\partial_{\bar{w}}\bar{\lambda}\right) - 2\frac{\sin\mu}{\cos^2\mu}\partial_w\lambda\,\partial_{\bar{w}}\bar{\lambda}\,\cos\vartheta = 0 \qquad (1.3)$$

Here ϑ is the field of the integrable system, λ is any holomorphic function of the complex coordinate w, and μ is a real harmonic function defined by $i\mu = \lambda - \bar{\lambda}$. The field ϑ is simply related to the dilaton by $e^{2i\vartheta} = \text{sh}(2\phi + 2\lambda)/\text{sh}(2\phi + 2\bar{\lambda})$. The equation (1.3) is invariant under conformal reparametrizations, just as Liouville theory is. Choosing the conformal coordinate to coincide with λ gives a non-translation-invariant equation, akin to Liouville theory in a non-translation invariant ground state, as was examined in [32, 33].

Remarkably, the system (1.3) is completely integrable. Actually, even better, it may be mapped onto a linear equation which can be solved exactly, and whose general solution may be exhibited in explicit form, just as in Liouville theory [32].

The remainder of this paper is organized as follows. In section 2, the interface Yang-Mills theory with the maximal number of 16 supersymmetries, and in section 3, Type IIB supergravity are briefly reviewed, mostly to fix notations. In section 4, the $AdS_4 \times S^2 \times S^2 \times \Sigma$ Ansatz is implemented on all the Type IIB supergravity fields, and in section 5, the BPS equations are reduced on this Ansatz. This reduction was already carried out by Gomis and Römelsberger [31]; the derivation given here is included in order to clarify a number of important issues and to give the proper S-duality interpretation of the reality conditions which are key to obtaining a full solution to the BPS equations.

In section 6, it is shown that every Type IIB solution with 16 supersymmetries may be mapped, using the $SL(2, \mathbf{R})$ S-duality of Type IIB supergravity, onto a solution in which the axion vanishes, and the 3-form field strengths, as well as the supersymmetry generating spinors obey certain reality conditions. In section 7, it is shown that the fully reduced BPS equations consist of two first order differential equations for the dilaton ϕ and the Weyl factor ρ of the metric on Σ, as well as two arbitrary holomorphic functions on Σ. It is further shown that this system of differential equations is automatically integrable. In section 8, the Bianchi identities and field equations are reduced to the $AdS_4 \times S^2 \times S^2 \times \Sigma$ Ansatz, and are shown to hold whenever ϕ and ρ are solutions to the BPS system of first order equations.

In section 9, a first change of variables is used to map the system onto the integrable system (1.3), for which the reduced BPS system constitutes a Bäcklund pair. A second change of variables is used to map this integrable system onto a set of linear equations, which is then solved in terms of two holomorphic functions, or equivalently, two harmonic functions h_1 and h_2, on Σ. The exact solution for the dilaton ϕ, the metric ρ, as well as all the other geometrical data entering the solution are obtained explicitly. In section 10, the Janus solution with 16 supersymmetries is identified and shown to be everywhere regular.

The general solutions obtained in this paper will be the starting point in a companion paper [1] for the construction of infinite classes of non-singular solutions corresponding to back-reacted solutions of $AdS_5 \times S^5$ and Janus doped with D5 and NS5 branes. These solutions generalize the supersymmetric Janus solution found in this paper. Instead of two there can be $2g+2$ asymptotic $AdS_5 \times S^5$ regions where the dilaton approaches (in general) different values. In addition, there are non-trivial NSNS and RR 3-form fluxes present in these solutions. In certain limits the geometry has singularities which correspond to probe D5 and NS5 branes. The AdS/CFT duals correspond to generalized interface Yang-Mills theories.

There is a closely related supergravity solution which has $SO(2,1) \times SO(3) \times SO(5)$ symmetry and is described by an $AdS_2 \times S^2 \times S^4 \times \Sigma$ Ansatz. The gravitational solution describes the fully back reacted geometry dual to half-BPS Wilson loops [34–36]. A detailed analysis of this solution applying the methods of this paper can be found in a further companion paper [37].

2. Interface Yang-Mills with maximal supersymmetry

The Yang-Mills theory with planar interface and maximal supersymmetry has 8 Poincaré supersymmetries, an additional 8 supersymmetries in the conformal limit, and $SO(3) \times SO(3) \sim SU(2) \times SU(2)$ R-symmetry. This reduced R-symmetry canonically splits the scalar multiplet into two triplets, which we shall denote by ϕ^i and $\tilde{\phi}^i$, with $i \in \{1,3,5\}$ for ϕ^i and $i \in \{2,4,6\}$ for $\tilde{\phi}^i$. Under $SO(3) \times SO(3)$ the triplet ϕ transforms as $(\mathbf{3}, \mathbf{1})$, while $\tilde{\phi}$ transforms as $(\mathbf{1}, \mathbf{3})$. The bulk Lagrangian is given by

$$
\begin{aligned}
\mathcal{L}_{\text{bulk}} = {} & -\frac{1}{4g^2}\text{tr}\left(F^{\mu\nu}F_{\mu\nu}\right) - \frac{1}{2g^2}\text{tr}(D^\mu\phi^i D_\mu\phi^i) - \frac{g^2}{2}\text{tr}(D^\mu\tilde{\phi}^i D_\mu\tilde{\phi}^i) \\
& + \frac{1}{4g^2}\text{tr}([\phi^i,\phi^j][\phi^i,\phi^j]) + \frac{g^2}{2}\text{tr}([\tilde{\phi}^i,\phi^j][\tilde{\phi}^i,\phi^j]) + \frac{g^6}{4}\text{tr}([\tilde{\phi}^i,\tilde{\phi}^j][\tilde{\phi}^i,\tilde{\phi}^j]) \\
& - \frac{i}{2g^2}\text{tr}\left(\bar{\psi}\gamma^\mu D_\mu\psi\right) + \frac{i}{2g^2}\text{tr}\left(D_\mu\bar{\psi}\gamma^\mu\psi\right) + \frac{1}{2g^2}\text{tr}\left(\psi^t\mathcal{C}\rho^i[\phi^i,\psi] + \psi^\dagger\mathcal{C}(\rho^i)^*[\phi^i,\psi^*]\right) \\
& + \frac{1}{2}\text{tr}\left(\psi^t\mathcal{C}\rho^i[\tilde{\phi}^i,\psi] + \psi^\dagger\mathcal{C}(\rho^i)^*[\tilde{\phi}^i,\psi^*]\right)
\end{aligned}
\tag{2.1}
$$

and the interface Lagrangian is given by

$$
\mathcal{L}_{\text{interface}} = \frac{(\partial_\pi g)}{g^3}\text{tr}\left(\frac{i}{2}\psi^t\mathcal{C}\psi + \frac{i}{2}\psi^\dagger\mathcal{C}\psi^* - \frac{2}{3}ig^6\epsilon^{ijk}\tilde{\phi}^i[\tilde{\phi}^j,\tilde{\phi}^k]\right)
\tag{2.2}
$$

Here, the Yang-Mills coupling $g(x^\pi)$ is a function of the coordinate x^π transverse to the interface. The interface theory which is AdS/CFT dual to the supersymmetric Janus solution has conformal symmetry, achieved by choosing $g(x^\pi)$ to be a step function. For this choice, the interface term (2.2) is localized at $x^\pi = 0$ and the superconformal symmetry respecting the location of the interface is restored (for notation and details, see [7]).

Notice that, in both the bulk and interface Lagrangians, the scalar triplets ϕ and $\tilde{\phi}$ enter with different scalings of the gauge coupling g.[2] The space of inter-

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[2]The bulk Lagrangian may be put in a more standard from by scaling the scalar fields as $\tilde{\phi}^i \to g^{-2}\tilde{\phi}^i$ at the cost of introducing interface operators of the form $(\partial_\pi g)^2\tilde{\phi}^i\tilde{\phi}^j$.

face theories is parametrized by the gauge coupling and the interface couplings $U \in$ SO(6)/(SO(3) × SO(3)), which rotate the embedding of SO(3) × SO(3) in SO(6). Theories for different U are physically equivalent, although described by a different set of couplings.

The interface theories are different in character from the defect CFT discussed in the AdS/CFT context in [8, 9, 11 – 13]. In an interface theory there are no new degrees of freedom (e.g. hypermultiplets coming from open strings localized at brane intersections) living on the interface other than the ones already present in the bulk.

3. Type IIB supergravity

For completeness, we briefly review the Type IIB supergravity Bianchi identities and field equations, as well as the supersymmetry variations, all for vanishing fermion fields. Our conventions are those of [24, 38] (see also [39]). The bosonic fields are: the metric g_{MN}; the complex axion-dilaton scalar B; the complex 2-form $B_{(2)}$ and the real 4-form $C_{(4)}$. We introduce composite fields in terms of which the field equations are expressed simply, as follows,

$$
\begin{aligned}
P &= f^2 dB & f^2 &= (1 - |B|^2)^{-1} \\
Q &= f^2 \mathrm{Im}(B d\bar{B})
\end{aligned}
\tag{3.1}
$$

and the field strengths $F_{(3)} = dB_{(2)}$, and

$$
\begin{aligned}
G &= f(F_{(3)} - B\bar{F}_{(3)}) \\
F_{(5)} &= dC_{(4)} + \frac{i}{16} \left(B_{(2)} \wedge \bar{F}_{(3)} - \bar{B}_{(2)} \wedge F_{(3)} \right)
\end{aligned}
\tag{3.2}
$$

The scalar field B is related to the complex string coupling τ, the axion χ, and dilaton Φ (for notational convenience we use $\phi \equiv \Phi/2$ for the dilaton field) by

$$
B = \frac{1 + i\tau}{1 - i\tau} \qquad\qquad \tau = \chi + ie^{-2\phi}
\tag{3.3}
$$

In terms of the composite fields P, Q, and G, there are Bianchi identities given as follows,

$$
0 = dP - 2iQ \wedge P
\tag{3.4}
$$

$$
0 = dQ + iP \wedge \bar{P}
\tag{3.5}
$$

$$
0 = dG - iQ \wedge G + P \wedge \bar{G}
\tag{3.6}
$$

$$
0 = dF_{(5)} - \frac{i}{8} G \wedge \bar{G}
\tag{3.7}
$$

The field strength $F_{(5)}$ is required to be self-dual,

$$
F_{(5)} = *F_{(5)}
\tag{3.8}
$$

The field equations are given by,

$$0 = \nabla^M P_M - 2iQ^M P_M + \frac{1}{24} G_{MNP} G^{MNP} \tag{3.9}$$

$$0 = \nabla^P G_{MNP} - iQ^P G_{MNP} - P^P \bar{G}_{MNP} + \frac{2}{3} i F_{(5)MNPQR} G^{PQR} \tag{3.10}$$

$$0 = R_{MN} - P_M \bar{P}_N - \bar{P}_M P_N - \frac{1}{6} (F_{(5)}^2)_{MN}$$
$$- \frac{1}{8} (G_M{}^{PQ} \bar{G}_{NPQ} + \bar{G}_M{}^{PQ} G_{NPQ}) + \frac{1}{48} g_{MN} G^{PQR} \bar{G}_{PQR} \tag{3.11}$$

The fermionic fields are the dilatino λ and the gravitino ψ_M, both of which are complex Weyl spinors with opposite 10-dimensional chiralities, given by $\Gamma_{11} \lambda = \lambda$, and $\Gamma_{11} \psi_M = -\psi_M$. The supersymmetry variations of the fermions are[3]

$$\delta\lambda = i(\Gamma \cdot P)\mathcal{B}^{-1} \varepsilon^* - \frac{i}{24} (\Gamma \cdot G)\varepsilon \tag{3.12}$$

$$\delta\psi_M = D_M \varepsilon + \frac{i}{480} (\Gamma \cdot F_{(5)})\Gamma_M \varepsilon - \frac{1}{96} \left(\Gamma_M (\Gamma \cdot G) + 2(\Gamma \cdot G)\Gamma^M \right) \mathcal{B}^{-1} \varepsilon^*$$

where $\mathcal{B}$ is the charge conjugation matrix of the Clifford algebra.[4] The BPS equations are obtained by setting $\delta\lambda = \delta\psi_M = 0$.

3.1 SU(1,1) symmetry

Type IIB supergravity is invariant under $\mathrm{SU}(1,1) \sim \mathrm{SL}(2,\mathbf{R})$ symmetry, which leaves $g_{\mu\nu}$ and $C_{(4)}$ invariant, acts by Möbius transformation on the field B, and linearly on $B_{(2)}$,

$$B \to B^s = \frac{uB + v}{\bar{v}B + \bar{u}}$$
$$B_{(2)} \to B_{(2)}^s = uB_{(2)} + v\bar{B}_{(2)} \tag{3.13}$$

with $u, v \in \mathbf{C}$ and $\bar{u}u - \bar{v}v = 1$. In this non-linear realization of $\mathrm{SU}(1,1)$, the field B takes values in the coset $\mathrm{SU}(1,1)/\mathrm{U}(1)_q$, and the fermions λ and ψ_μ transform linearly under the isotropy gauge group $\mathrm{U}(1)_q$ with composite gauge field Q. The transformation rules for the composite fields are [24],

$$P \to P^s = e^{2i\theta} P$$
$$Q \to Q^s = Q + d\theta$$
$$G \to G^s = e^{i\theta} G \tag{3.14}$$

where the phase θ is defined by

$$e^{i\theta} = \left(\frac{v\bar{B} + u}{\bar{v}B + \bar{u}} \right)^{\frac{1}{2}} \tag{3.15}$$

In this form, the transformation rules clearly exhibit the $\mathrm{U}(1)_q$ gauge transformation that accompanies the global $\mathrm{SU}(1,1)$ transformations.

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[3]Throughout, we shall use the notation $\Gamma \cdot T \equiv \Gamma^{M_1 \cdots M_p} T_{M_1 \cdots M_p}$ for the contraction of any antisymmetric tensor field T of rank p and the Γ-matrix of the same rank.

[4]It is defined by $\mathcal{B}\mathcal{B}^* = I$ and $\mathcal{B}\Gamma^M \mathcal{B}^{-1} = (\Gamma^M)^*$; see appendix A for our Γ-matrix conventions. Throughout, complex conjugation will be denoted by *bar* for functions, and by *star* for spinors.

4. The two-parameter Ansatz

We seek a general Ansatz in Type IIB supergravity with the following symmetry,

$$\text{SO}(2,3) \times \text{SO}(3) \times \text{SO}(3) \tag{4.1}$$

which may be viewed as the bosonic subgroup of $OSp(2,2|4)$. The factor $\text{SO}(2,3)$ requires the geometry to contain AdS_4, while the factor $\text{SO}(3) \times \text{SO}(3)$ could be accommodated by either $S^2 \times S^2$ or S^3.

Given that our initial motivation was the construction of a Janus solution with 16 supersymmetries, and dual to the interface Yang-Mills theory with maximal supersymmetry of section 2, the case of S^3 is excluded. This is because the 6 scalar fields ϕ^i and $\tilde{\phi}^j$, with $i = 1, 3, 5$ and $j = 2, 4, 6$ are grouped in two independent sets which immediately suggests $S^2 \times S^2$. Two dimensions remain undetermined by the symmetries alone, so that the most general space of interest to us will be of the form,

$$AdS_4 \times S_1^2 \times S_2^2 \times \Sigma \tag{4.2}$$

where Σ stands for the two-dimensional space, over which the above products are warped. In order for the above space to be a Type IIB supergravity geometry, Σ must carry an orientation as well as a Riemannian metric, and is therefore a Riemann surface, generally with boundary. The subscripts 1 and 2 label the two-spheres.

4.1 Ansatz for the type IIB fields

The Ansatz for the metric is

$$ds^2 = f_4^2 ds_{AdS_4}^2 + f_1^2 ds_{S_1^2}^2 + f_2^2 ds_{S_2^2}^2 + ds_\Sigma^2 \tag{4.3}$$

where f_1, f_2, f_4 and ds_Σ^2 are functions on Σ. We introduce an orthonormal frame,

$$
\begin{aligned}
e^m &= f_4 \, \hat{e}^m & m &= 0, 1, 2, 3 \\
e^{i_1} &= f_1 \, \hat{e}^{i_1} & i_1 &= 4, 5 \\
e^{i_2} &= f_2 \, \hat{e}^{i_2} & i_2 &= 6, 7 \\
e^a & & a &= 8, 9
\end{aligned}
\tag{4.4}
$$

where $\hat{e}^m$, $\hat{e}^{i_1}$, $\hat{e}^{i_2}$ and e^a refer to orthonormal frames for the spaces AdS_4, S_1^2, S_2^2 and Σ respectively. In particular, we have[5]

$$
\begin{aligned}
ds_{AdS_4}^2 &= \eta_{mn} \, \hat{e}^m \otimes \hat{e}^n \\
ds_{S_1^2}^2 &= \delta_{i_1 j_1} \, \hat{e}^{i_1} \otimes \hat{e}^{j_1} \\
ds_{S_2^2}^2 &= \delta_{i_2 j_2} \, \hat{e}^{i_2} \otimes \hat{e}^{j_2} \\
ds_\Sigma^2 &= \delta_{ab} \, e^a \otimes e^b
\end{aligned}
\tag{4.5}
$$

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[5]The convention of summation over repeated indices will be used throughout whenever no confusion is expected to arise, with the ranges of the various indices following the pattern of the frame in (4.4).

where $\eta = \mathrm{diag}[-+++]$. The complex dilaton/axion field P, and the connection Q are 1-forms, and their structure is simply given as follows,

$$P = p_a e^a$$
$$Q = q_a e^a \tag{4.6}$$

Throughout, we shall view the P and Q-forms as given in terms of the dilaton/axion field B, as in (3.1), so that P and Q are not independent fields. Thus, they will always automatically satisfy their Bianchi identities,

$$dP - 2iQ \wedge P = 0$$
$$dQ + iP \wedge \bar{P} = 0 \tag{4.7}$$

This approach will allow us to dispense with the Q field and show that every half-BPS solution in fact arises as a $SU(1,1)$ transformation of a solution with vanishing axion.

Finally, the anti-symmetric tensor forms $F_{(5)}$ and G are given by

$$F_{(5)} = f_a(-e^{0123a} + \varepsilon^a{}_b\, e^{4567b}) \qquad\qquad \varepsilon^{89} = 1$$
$$G = g_a e^{45a} + ih_a e^{67a} \tag{4.8}$$

Here, f_a, q_a are real, while g_a, h_a, p_a are complex. It will be useful to introduce 1-forms for these reduced fields as well,

$$\mathcal{F} \equiv f_a e^a \qquad\qquad *_2\,\mathcal{F} = \varepsilon^a{}_b\, f_a\, e^b$$
$$\mathcal{G} \equiv g_a e^a$$
$$\mathcal{H} \equiv h_a e^a \tag{4.9}$$

so that we have equivalently,

$$F_{(5)} = -e^{0123} \wedge \mathcal{F} + e^{4567} \wedge *_2\mathcal{F}$$
$$G = e^{45} \wedge \mathcal{G} + ie^{67} \wedge \mathcal{H} \tag{4.10}$$

Here, $*_2$ denotes the Poincaré dual on Σ with respect to the metric ds_Σ^2.

5. Reduced BPS equations with 16 supersymmetries

Solutions of the form given by the Ansatz of subsection 4.1 which preserve 16 supersymmetries correspond to supergravity fields for which the BPS equations $\delta\lambda = \delta\psi_M = 0$ in (3.12) have 16 independent solutions ε. Whenever the dilaton is subject to a non-trivial space-time variation, $\partial_M\phi \neq 0$, the dilatino BPS equation will allow for at most 16 independent supersymmetries ε. Therefore, the gravitino BPS equation should not impose any further restrictions on the number of supersymmetries, but should instead simply give the space-time evolution of ε. As a result, at any given point in the space Σ, ε must be a Killing spinor on each of the spheres $S_1^2 \times S_2^2$, as well as on AdS_4.

The analysis in this section is similar to the one employed in [31], and we use a closely related notation. The method of bilinears in the Killing spinors, pioneered in [40], is not needed here, and the corresponding results will be derived systematically from the reduced BPS equations instead. To illustrate our method, and for the sake of additional clarity and completeness, the derivation will be presented here in detail.

5.1 Using Killing spinors

Killing spinors on $AdS_4 \times S_1^2 \times S_2^2$ are non-vanishing solutions to the equations,

$$
\left(\hat{\nabla}_m - \frac{1}{2} \eta_1 \gamma_m \otimes I_2 \otimes I_2 \right) \chi_{\eta_0}^{\eta_1, \eta_2, \eta_3} = 0 \qquad\qquad m = 0, 1, 2, 3
$$

$$
\left(\hat{\nabla}_{i_1} - \frac{i}{2} \eta_2 I_4 \otimes \gamma_{i_1} \otimes I_2 \right) \chi_{\eta_0}^{\eta_1, \eta_2, \eta_3} = 0 \qquad\qquad i = 4, 5
$$

$$
\left(\hat{\nabla}_{i_2} - \frac{i}{2} \eta_3 I_4 \otimes I_2 \otimes \gamma_{i_2} \right) \chi_{\eta_0}^{\eta_1, \eta_2, \eta_3} = 0 \qquad\qquad i = 6, 7 \tag{5.1}
$$

Our conventions for the Clifford algebra are given in appendix A. The spinors $\chi_{\eta_0}^{\eta_1, \eta_2, \eta_3}$ are 16-dimensonal. The covariant derivatives $\hat{\nabla}_m$, $\hat{\nabla}_{i_1}$, and $\hat{\nabla}_{i_2}$ act in the Dirac spinor representations for AdS_4, S_1^2, and S_2^2, with respect to the canonical spin connections associated with the frames $\hat{e}^m$, $\hat{e}^{i_1}$ and $\hat{e}^{i_2}$. Once the integrability conditions, $\eta_1^2 = \eta_2^2 = \eta_3^2 = 1$, for (5.1) are satisfied, the solution spaces are of maximal dimension, namely 16.

Since the chirality matrix for each Killing spinor equation (respectively $\gamma_{(1)}$, $\gamma_{(2)}$, and $\gamma_{(3)}$) commutes with the corresponding covariant derivative, but not with the entire equation in (5.1), the chirality matrices will map between two linearly independent solutions. This is explained in detail in appendix B, where the geometry of Killing spinors on S^2 and AdS_4 is reviewed. Thus, we may use η_1, η_2, η_3 to label the linearly independent solutions to the Killing spinor equations. The Killing equation for AdS_4, however, has 4 linearly independent solutions, and the label η_1 alone does not suffice to label these solutions uniquely. The 4 solutions consist of 2 degenerate solutions for each chirality, and this degeneracy is uniquely specified by the extra label $\eta_0 = \pm 1$. In total, the solutions of the AdS_4 Killing equation are uniquely labeled by the pair (η_0, η_1). To economize notation, the index η_0 will be not be exhibited, with the understanding that the solution space for $\chi^{\eta_1, \eta_2, \eta_3}$ remains 16-dimensional.

For any one of the chirality matrices $\gamma_{(s)}$, for $s = 1, 2, 3$, the product $\gamma_{(s)} \chi$ satisfies (5.1) with the opposite value of η_s. We may therefore identify the corresponding spinors,

$$
\gamma_{(1)} \chi^{\eta_1, \eta_2, \eta_3} = \chi^{-\eta_1, \eta_2, \eta_3}
$$

$$
\gamma_{(2)} \chi^{\eta_1, \eta_2, \eta_3} = \chi^{\eta_1, -\eta_2, \eta_3}
$$

$$
\gamma_{(3)} \chi^{\eta_1, \eta_2, \eta_3} = \chi^{\eta_1, \eta_2, -\eta_3} \tag{5.2}
$$

To examine the Killing spinor properties, we begin by decomposing the 32 component (complex) spinor ε onto the Σ-independent basis of spinors $\chi_{\eta_0}^{\eta_1, \eta_2, \eta_3}$, with coefficients which are Σ-dependent 2-component spinors $\zeta_{\eta_1, \eta_2, \eta_3}$,

$$
\varepsilon = \sum_{\eta_1, \eta_2, \eta_3} \chi^{\eta_1, \eta_2, \eta_3} \otimes \zeta_{\eta_1, \eta_2, \eta_3} \tag{5.3}
$$

The 10-dimensional chirality condition $\Gamma^{11}\varepsilon = -\varepsilon$ reduces to

$$\gamma_{(4)}\zeta_{-\eta_1,-\eta_2,-\eta_3} = -\zeta_{\eta_1,\eta_2,\eta_3} \tag{5.4}$$

where $\gamma_{(4)}$ is the chirality matrix associated with Σ; see appendix D for its detailed expression. The Killing spinor equations are invariant under charge conjugation $\chi \to \chi^c$, with

$$(\chi^c)^{\eta_1,\eta_2,\eta_3} = B_{(1)} \otimes B_{(2)} \otimes B_{(3)} \, (\chi^{\eta_1,\eta_2,\eta_3})^* \tag{5.5}$$

where $B_{(1)}$, $B_{(2)}$, $B_{(3)}$ are the charge conjugation matrices on the Dirac algebras for AdS_4, S_1^2 and S_2^2 respectively. Since $(B_{(1)} \otimes B_{(2)} \otimes B_{(3)})^2 = I_{16}$, we may impose, without loss of generality, the reality condition $\chi^c = \pm\chi$ on the basis. The sign assignments are related by (5.2), and found to be

$$B_{(1)} \otimes B_{(2)} \otimes B_{(3)} \, (\chi^{\eta_1,\eta_2,\eta_3})^* = \eta_1\eta_2\eta_3\chi^{\eta_1,\eta_2,\eta_3} \tag{5.6}$$

Upon imposing the reality condition (5.6) on the basis of spinors χ, and the chirality condition (5.4) on ζ, and recalling that $\chi^{\eta_1\eta_2\eta_3}$ has double degeneracy due to the suppressed quantum number η_0, we indeed recover 16 complex components for the spinor ε.

Following [31], we introduce a matrix notation in the 8-dimensional space of η by,

$$\tau^{(ijk)} \equiv \tau^i \otimes \tau^j \otimes \tau^k \qquad\qquad i,j,k = 0,1,2,3 \tag{5.7}$$

where $\tau^0 = I_2$, and τ^i with $i = 1,2,3$ are the Pauli matrices in the standard basis. Multiplication by $\tau^{(ijk)}$ is defined as follows,

$$(\tau^{(ijk)}\zeta)_{\eta_1,\eta_2,\eta_3} \equiv \sum_{\eta_1',\eta_2',\eta_3'} (\tau^i)_{\eta_1\eta_1'}(\tau^j)_{\eta_2\eta_2'}(\tau^k)_{\eta_3\eta_3'}\zeta_{\eta_1',\eta_2',\eta_3'} \tag{5.8}$$

Henceforth, we shall use matrix notation for τ and suppress the indices η.

5.2 The reduced BPS equations

With the help of the Ansatz for the Type IIB fields produced in subsection 4.1, the BPS equations (3.12) may be reduced and presented using the notations introduced in the preceding subsection. The explicit reduction is carried out in appendix C. The dilatino BPS equation is given by,

$$(d) \qquad 0 = p_a\sigma^a\sigma^2\zeta^* + \frac{1}{4}\left(g_a\tau^{(322)} - ih_a\tau^{(333)}\right)\sigma^a\zeta \tag{5.9}$$

while the gravitino equation decomposes into a system of 4 equations,

$$(m) \qquad 0 = -\frac{i}{2f_4}\tau^{(211)}\zeta + \frac{D_af_4}{2f_4}\sigma^a\zeta + \frac{1}{2}f_a\tau^{(100)}\sigma^a\zeta + \frac{1}{16}\left(g_a\tau^{(322)} - ih_a\tau^{(333)}\right)\sigma^a\sigma^2\zeta^*$$

$$(i_1) \qquad 0 = \frac{1}{2f_1}\tau^{(021)}\zeta + \frac{D_af_1}{2f_1}\sigma^a\zeta - \frac{1}{2}f_a\tau^{(100)}\sigma^a\zeta + \frac{1}{16}\left(-3g_a\tau^{(322)} - ih_a\tau^{(333)}\right)\sigma^a\sigma^2\zeta^*$$

$$(i_2) \qquad 0 = \frac{1}{2f_2}\tau^{(002)}\zeta + \frac{D_af_2}{2f_2}\sigma^a\zeta - \frac{1}{2}f_a\tau^{(100)}\sigma^a\zeta + \frac{1}{16}\left(g_a\tau^{(322)} + 3ih_a\tau^{(333)}\right)\sigma^a\sigma^2\zeta^*$$

$$(a) \qquad 0 = D_a\zeta + \frac{i}{2}\hat{\omega}_a\sigma^3\zeta - \frac{i}{2}q_a\zeta + \frac{1}{2}f_b\tau^{(100)}\sigma^b\sigma_a\zeta + \frac{1}{16}\left(-3g_a\tau^{(322)} + g_b\tau^{(322)}\sigma^{ab}\right)\sigma^2\zeta^*$$
$$+ \frac{1}{16}\left(3ih_a\tau^{(333)} - ih_b\tau^{(333)}\sigma^{ab}\right)\sigma^2\zeta^* \tag{5.10}$$

The derivatives D_a are defined with respect to the frame e^a, so that $e^a D_a = d$, the total differential on Σ. Also, we denote the Dirac matrices γ^a on Σ simply by σ^a, in a slight abuse of notation where $\sigma^8 = \sigma^1$ and $\sigma^9 = \sigma^2$, in accord with the conventions of appendix A.

5.3 Symmetries of the reduced BPS equations

The reduced BPS equations exhibit continuous as well as discrete symmetries, which will be exploited to further reduce them. The continuous symmetries are as follows.

Local frame rotations of the frame e^a on Σ generate a gauge symmetry $\mathrm{U}(1)_c$, whose action on all fields is standard.

The axion/dilaton field B transforms non-linearly under the continuous S-duality group $\mathrm{SU}(1,1)$ of Type IIB supergravity. As was discussed at the end of section 3, B takes values in the coset $\mathrm{SU}(1,1)/\mathrm{U}(1)_q$, and $\mathrm{SU}(1,1)$ transformations on the fields are accompanied by local $\mathrm{U}(1)_q$ gauge transformations, given in (3.14) and (3.15),

$$
U(1)_q \qquad
\begin{aligned}
\zeta &\to e^{i\theta/2}\zeta \\
q_a &\to q_a + D_a\theta \\
p_a &\to e^{2i\theta}p_a \\
g_a &\to e^{i\theta}g_a \\
h_a &\to e^{i\theta}h_a
\end{aligned}
\tag{5.11}
$$

The real function θ depends on the $\mathrm{SU}(1,1)$ transformation, as well as on the field B.

5.3.1 Discrete symmetries

The reduced BPS equations are also invariant under three commuting involutions. The first two do not mix ζ and ζ^* and leave the fields f_a, p_a, q_a, g_a, h_a unchanged. They are defined by,

$$
\begin{aligned}
\mathcal{I}\zeta &= -\tau^{(111)}\sigma^3\zeta \\
\mathcal{J}\zeta &= \tau^{(032)}\zeta
\end{aligned}
\tag{5.12}
$$

Both $\mathcal{I}$ and $\mathcal{J}$ commute with the symmetries $\mathrm{U}(1)_q$ and $\mathrm{U}(1)_c$.

5.3.2 Complex conjugation

The third involution $\mathcal{K}$ amounts to complex conjugation. This operation acts non-trivially on all complex fields, and its action on ζ depends on the basis of τ-matrices. In a basis in which both σ^2 and τ^2 are purely imaginary, the involution $\mathcal{K}$ has the following form. Taking the complex conjugates of p_a, g_a, h_a, letting $q_a \to -q_a$ and mapping $\zeta \to -i\,\tau^{(020)}\sigma^2\zeta^*$ will leave the BPS equations invariant.

Complex conjugation, defined this way, however, does not commute with the $\mathrm{SU}(1,1)$ transformations, since ζ transforms under $\mathrm{SU}(1,1)$ by a local $\mathrm{U}(1)_q$ gauge transformation.

Therefore, we relax the previous definition of complex conjugation, and allow for complex conjugation modulo a $\mathrm{U}(1)_q$ gauge transformation with phase θ,

$$
\begin{aligned}
\mathcal{K}\zeta &= e^{i\theta}\tau^{(020)}\sigma^2\zeta^* \\
\mathcal{K}q_a &= -q_a + 2D_a\theta \\
\mathcal{K}p_a &= +e^{4i\theta}\bar{p}_a \\
\mathcal{K}g_a &= -e^{2i\theta}\bar{g}_a \\
\mathcal{K}h_a &= -e^{2i\theta}\bar{h}_a
\end{aligned}
\tag{5.13}
$$

which continues to be a symmetry of the BPS equations.[6] The need for such a compensating gauge transformation should be clear from the fact that ζ and ζ^* transform with opposite phases under $\mathrm{U}(1)_q$. On the other hand, $\mathcal{K}$ commutes with the group $\mathrm{U}(1)_c$ of frame rotations.

5.3.3 Restricting chirality in Type IIB

In Type IIB theory only a single chirality is retained, so we have the condition

$$
\mathcal{I}\zeta = -\tau^{(111)}\sigma^3\zeta = \zeta
\tag{5.14}
$$

This subspace is invariant under the remaining involutions, since $\mathcal{J}$ and $\mathcal{K}$ commute with $\mathcal{I}$.

6. Reality properties of the supersymmetric solution

In this section, we shall establish that the BPS equations restrict ζ to belong to a single one of the eigenspaces of $\mathcal{J}$, but not both, and to a single one of the eigenspaces of $\mathcal{K}$, but not both. These results lead to a further reduction of the BPS equations.[7] In particular, we shall show that every solution with 16 supersymmetries may be mapped by an $\mathrm{SU}(1,1)$ transformation onto a solution with vanishing axion field and real g_a, h_a. The Janus solution with 4 supersymmetries, obtained in [24], exhibits an analogous reality property.

The restrictions of ζ to definite eigenspaces of $\mathcal{J}$ and $\mathcal{K}$ may be established directly from the BPS equations, by showing that they imply a certain number of *bilinear constraints* on ζ (and ζ^*) which are independent of reduced fields $f_1, f_2, f_4, f_a, g_a, h_a, p_a, q_a$.

6.1 Restriction to a single eigenspace of $\mathcal{J}$

The restriction for $\mathcal{J}$ is obtained as follows; the detailed arguments are presented in appendix D. Contracting the dilatino BPS equation (d) of (5.9) on the left by $\zeta^\dagger T\sigma^{0,3}$ for certain τ matrices T, and using the assumption $\partial_a B \neq 0$, leads to a first set of constraints,

$$
\zeta^\dagger T\sigma^a\zeta = 0 \qquad\qquad T \in \mathcal{T} = \left\{\tau^{(310)},\ \tau^{(301)},\ \tau^{(201)},\ \tau^{(210)}\right\}
\tag{6.1}
$$

---

[6]The factor of i in the transformation rule for ζ has been absorbed into the compensating $\mathrm{U}(1)_q$ transformation.

[7]In [31], this further reduction was achieved upon the additional use of the closure of the supersymmetry algebra. Here, it is shown that this is in fact unnecessary and that the entire further reduction of the BPS equations follows directly from the BPS equations themselves.

and $a = 1, 2$. Contracting the gravitino BPS equations (m), (i_1) and (i_2) of (5.10) on the left by $\zeta^\dagger T \sigma^{0,3}$, with $T \in \mathcal{T}$, and using the vanishing of terms involving g_a, h_a due to (6.1), leads to a second set of constraints,

$$\zeta^\dagger U \sigma^p \zeta = 0 \qquad U \in \mathcal{U} = \Big\{ \tau^{(001)}, \tau^{(010)}, \tau^{(231)}, \tau^{(220)}, \tau^{(212)}, \tau^{(203)},$$
$$\tau^{(110)}, \tau^{(101)}, \tau^{(320)}, \tau^{(331)}, \tau^{(303)}, \tau^{(312)} \Big\} \quad (6.2)$$

for $p = 0, 3$. In appendix D, a detailed derivation of the solution to both sets of bilinear constraints is given. The general solution may be expressed as the projection condition onto a single eigenspace of $\mathcal{J}$,

$$\mathcal{J}\zeta = \tau^{(032)}\zeta = \nu\zeta \tag{6.3}$$

where ν is either $+1$ or -1. The constraints (6.1) and (6.2) are automatically satisfied once (6.3) is, since $\tau^{(032)}T$ and $\tau^{(032)}U$ anticommute with the chirality constraint (5.14).

6.2 Restriction to a single eigenspace of $\mathcal{K}$

The use of the T-matrices in (6.1), and the U-matrices in (6.2), has allowed us to obtain relations between bilinears in ζ in which the dependence on both g_a and h_a was eliminated. Further useful information may be obtained from relations in which the dependence on either g_a or h_a, but not both, is eliminated. This is achieved by contracting the equations (m), (i_1) and (i_2) respectively by $\zeta^\dagger M_g \sigma^p$ and $\zeta^\dagger M_h \sigma^p$, where $p = 0, 3$, and the τ-matrices M_g and M_h are Hermitian and satisfy,

$$\left(M_g \tau^{(333)} \right)^t = +M_g \tau^{(333)} \qquad\qquad \left(M_h \tau^{(333)} \right)^t = -M_h \tau^{(333)}$$
$$\left(M_g \tau^{(322)} \right)^t = -M_g \tau^{(322)} \qquad\qquad \left(M_h \tau^{(322)} \right)^t = +M_h \tau^{(322)} \tag{6.4}$$

Non-trivial relations are obtained only if the corresponding matrices M_g and M_h commute with $\tau^{(032)}$, and if the products $M_g \tau^{(211)}$ and $M_h \tau^{(211)}$ commute with $\tau^{(111)}$. (It then follows that $M_g \tau^{(021)}$, $M_g \tau^{(002)}$, $M_h \tau^{(021)}$, and $M_h \tau^{(002)}$ commute with $\tau^{(111)}$.) Finally, using the restrictions on ζ under the involutions $\mathcal{I}$ and $\mathcal{J}$, we may consider M_g and M_h modulo equivalence under multiplication by $\tau^{(032)}$ and $\tau^{(111)}\sigma^3$. This leaves unique solutions,

$$M_g = \tau^{(002)}$$
$$M_h = \tau^{(021)} \tag{6.5}$$

We shall analyze the case M_h in detail, and simply quote the results from M_g. We start by multiplying the (m), (i_1), and (i_2) equations in (5.10) by $\zeta^\dagger \tau^{(021)}\sigma^p$, for $p = 0, 3$, to get

$$0 = -\frac{1}{2f_4}\zeta^\dagger \tau^{(230)}\sigma^p\zeta + \frac{\partial_a f_4}{2f_4}\zeta^\dagger \tau^{(021)}\sigma^p\sigma^a\zeta + \frac{1}{2}f_a\zeta^\dagger \tau^{(121)}\sigma^p\sigma^a\zeta + \frac{i}{16}g_a\zeta^\dagger \tau^{(303)}\sigma^p\sigma^a\sigma^2\zeta^*$$

$$0 = \frac{1}{2f_1}\zeta^\dagger \sigma^p\zeta + \frac{\partial_a f_1}{2f_1}\zeta^\dagger \tau^{(021)}\sigma^p\sigma^a\zeta - \frac{1}{2}f_a\zeta^\dagger \tau^{(121)}\sigma^p\sigma^a\zeta - \frac{3i}{16}g_a\zeta^\dagger \tau^{(303)}\sigma^p\sigma^a\sigma^2\zeta^* \quad (6.6)$$

$$0 = \frac{i}{2f_2}\zeta^\dagger \tau^{(023)}\sigma^p\zeta + \frac{\partial_a f_2}{2f_2}\zeta^\dagger \tau^{(021)}\sigma^p\sigma^a\zeta - \frac{1}{2}f_a\zeta^\dagger \tau^{(121)}\sigma^p\sigma^a\zeta + \frac{i}{16}g_a\zeta^\dagger \tau^{(303)}\sigma^p\sigma^a\sigma^2\zeta^*$$

For $p = 0$, the imaginary part of the first and second equations of (6.6), and for $p = 3$ the real part of the third equation of (6.6) obey the first two equations below,

$$\text{Im}\left(ig_a \zeta^\dagger \tau^{(303)} \sigma^a \sigma^2 \zeta^* \right) = 0$$

$$\text{Re}\left(ig_a \zeta^\dagger \tau^{(303)} \sigma^3 \sigma^a \sigma^2 \zeta^* \right) = 0$$

$$\text{Im}\left(h_a \zeta^\dagger \tau^{(331)} \sigma^a \sigma^2 \zeta^* \right) = 0$$

$$\text{Re}\left(h_a \zeta^\dagger \tau^{(331)} \sigma^3 \sigma^a \sigma^2 \zeta^* \right) = 0 \tag{6.7}$$

where the last two equations result from the analysis of the case M_g. Using the first line of (6.7) in the imaginary part of the third equation of (6.6) for $p = 0$, and the second line of (6.7) in the real parts of the first two equations of (6.6) for $p = 3$ gives the first three equations below,

$$\zeta^\dagger \tau^{(023)} \zeta = 0$$

$$\zeta^\dagger \sigma^3 \zeta = 0$$

$$\zeta^\dagger \tau^{(230)} \sigma^3 \zeta = 0$$

$$\zeta^\dagger \tau^{(213)} \sigma^3 \zeta = 0 \tag{6.8}$$

The analysis for M_g yields the first, second and fourth equations in (6.8).

It will be convenient to use the following rotated basis for the τ-matrices,

$$\tau^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau^3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{6.9}$$

Notice that the transposition and complex conjugation properties of these matrices are identical to those in the standard basis of Pauli matrices, so the equations (5.9) and (5.10) continue to hold unchanged in this basis.

The bilinear constraints of (6.8) are solved by the following complex conjugation relation,

$$\sigma^2 \zeta^* = e^{-i\theta} \tau^{(020)} \zeta \tag{6.10}$$

where θ is an arbitrary phase function on Σ, which is not fixed by (6.8). This result is easily verified by using (6.10) in the form $\zeta^\dagger = e^{-i\theta} \zeta^t \sigma^2 \tau^{(020)}$ to eliminate $\zeta^\dagger$ in (6.8) and then verifying that each of the four equations is of the form $\zeta^t M \zeta = 0$ with M anti-symmetric. In fact, one may check that (6.10) is the most general solution to (6.8), by decomposing ζ in components, and using the restrictions $\mathcal{I}\zeta = \zeta$ and $\mathcal{J}\zeta = \nu\zeta$. Equation (6.10) is just the condition that we restrict to $\mathcal{K}\zeta = \zeta$, i.e. to a single eigenspace of the involution $\mathcal{K}$.

6.3 Reality constraints

Having solved completely the bilinear constraints (6.8), it remains to solve (6.7). To do so, we use (6.10) to recast $\sigma^2 \zeta^*$ in terms of ζ, so that these constraints become,

$$\text{Im}\left(ie^{-i\theta} g_a\, \zeta^\dagger \tau^{(323)} \sigma^a \zeta \right) = 0$$

$$\text{Im}\left(ie^{-i\theta} g_a \, \varepsilon^{ab}\zeta^\dagger\tau^{(323)}\sigma_b\zeta\right) = 0$$

$$\text{Im}\left(ie^{-i\theta} h_a \, \zeta^\dagger\tau^{(311)}\sigma^a\zeta\right) = 0$$

$$\text{Im}\left(ie^{-i\theta} h_a \, \varepsilon^{ab}\zeta^\dagger\tau^{(311)}\sigma_b\zeta\right) = 0 \tag{6.11}$$

The bilinears $\zeta^\dagger\tau^{(323)}\sigma^a\zeta = \nu\zeta^\dagger\tau^{(311)}\sigma^a\zeta$ are real and non-vanishing, (the latter will be verified once the solution is obtained) so that

$$\text{Im}\left(ie^{-i\theta}g_a\right) = 0$$

$$\text{Im}\left(ie^{-i\theta}h_a\right) = 0 \qquad\qquad a = 1,2 \tag{6.12}$$

Finally, we perform the same elimination of $\zeta^\dagger$, using (6.10), also in the dilatino BPS equation (5.9) which, after multiplying through by $4\tau^{(020)}$, becomes,

$$4e^{-i\theta}p_a\sigma^a\zeta + g_a\tau^{(302)}\zeta + h_a\tau^{(313)}\zeta = 0 \tag{6.13}$$

Contracting to the left in turn by $\zeta^\dagger$ and $\zeta^\dagger\sigma^3$, and using (6.12), we obtain,

$$\text{Im}(p_a e^{-2i\theta}) = 0 \tag{6.14}$$

Here, we have assumed that $\zeta^\dagger\sigma^a\zeta$ does not identically vanish, as will be verified from the solution later on.

6.4 $SU(1,1)$ map to solutions with vanishing axion

Equation (6.14) implies that the dilaton/axion 1-form P satisfies $P = e^{2i\theta}\tilde{P}$, where $\tilde{P}$ is a real form. Using the second Bianchi identity in (4.7), it follows that $dQ = 0$, so that Q is pure gauge. Additionally, from the $SU(1,1)$ transformation laws (3.14) and (3.15), it follows that the phase θ is to be interpreted as the accompanying $U(1)_q$ gauge transformation of an $SU(1,1)$ transformation that maps the solution to the BPS equations onto a solution for which P is real, and $Q = 0$.

The fact that the BPS equations result in a reality condition on the supergravity fields and allow any solution to the BPS equations to be mapped onto a solution with vanishing axion (i.e. real P and $Q = 0$) is familiar from the study of the Janus problem with 4 supersymmetries in [24]), where an analogous result holds.

Performing now this $SU(1,1)$ transformation on all fields, we have $e^{-i\theta} = i$ (up to an immaterial choice of sign) so that the reality conditions become,

$$\bar{p}_a = p_a \qquad\qquad \bar{g}_a = g_a \qquad\qquad a = 8,9$$

$$q_a = 0 \qquad\qquad \bar{h}_a = h_a \tag{6.15}$$

Complex conjugation is now a symmetry with $\sigma^2\zeta^* = i\tau^{(020)}\zeta$.

6.5 The BPS equations reduced by $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ projections

In view of the involution constraints

$$
\begin{aligned}
\mathcal{I}\zeta &= -\tau^{(111)}\sigma^3\zeta = \zeta \\
\mathcal{J}\zeta &= \tau^{(032)}\zeta = \nu\zeta \\
\mathcal{K}\zeta &= -i\tau^{(020)}\sigma^2\zeta^* = \zeta
\end{aligned}
\tag{6.16}
$$

the non-vanishing components of ζ may be parametrized as follows,

$$
\begin{aligned}
\zeta_{+++-} &= +i\nu\,\zeta_{+---} = \alpha\,e^{i\theta/2} \\
\zeta_{---+} &= -i\nu\,\zeta_{-+++} = \bar{\beta}\,e^{i\theta/2} \\
\zeta_{+-++} &= +i\nu\,\zeta_{++-+} = \bar{\alpha}\,e^{i\theta/2} \\
\zeta_{-+--} &= -i\nu\,\zeta_{--+-} = \beta\,e^{i\theta/2}
\end{aligned}
\tag{6.17}
$$

Here, the first 3 indices on ζ refer to its η-assignments, and the last refers to its eigenvalue under σ^3, while $\alpha, \beta \in \mathbf{C}$. The overall constant phase is the one that resulted from the reality condition, and is given by $e^{-i\theta} = i$.

We shall analyze the BPS equations for a single chirality $\gamma_{(4)} = -1$; the opposite chirality equation is just the complex conjugate thereof. To do so, we use the complex frame[8] e^z and $e^{\bar{z}}$ on Σ. We begin by eliminating $\sigma^2\zeta^* = i\tau^{(020)}\zeta$ in favor of ζ in all equations of (5.9) and (5.10). To make the τ-matrices act simply, we make use of the relation $\tau^{(032)}\zeta = \nu\zeta$ in the first terms of equations (m) and (i_2) in (5.10), and recast them in the following form, $\tau^{(211)}\zeta = \nu\tau^{(223)}\zeta$ and $\tau^{(002)}\zeta = \nu\tau^{(030)}\zeta$.

The components of ζ may be regrouped in terms of their dependence on α and β in terms of spinors ξ and ξ^* whose $\gamma_{(4)}$-eigenvalues are -1 and $+1$ respectively,

$$
\xi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}
\qquad\qquad
\xi^* = \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix}
\tag{6.18}
$$

The τ-matrices in the basis (6.9) may be represented in terms of σ-matrices acting on the spinors ξ and ξ^* in the standard basis, according to the following rule,

$$
\begin{aligned}
\tau^{(011)} &= \pm\sigma^3 & \tau^{(223)} &= i\sigma^2 \\
\tau^{(302)} &= -\sigma^2 & \tau^{(021)} &= -i\sigma^3 \\
\tau^{(100)} &= \sigma^3 & \tau^{(030)} &= -\sigma^0
\end{aligned}
\tag{6.19}
$$

where $\pm$ refers to the action of the matrix on $\zeta_{\mp}$. The BPS equations, reduced by the

---

[8]Frame indices are defined by $e^z = (e^8 + ie^9)/2$ and $e^{\bar{z}} = (e^8 - ie^9)/2$, and are to be contracted with the flat Euclidean metric with non-zero components $\delta_{z\bar{z}} = \delta_{\bar{z}z} = 2$.

restrictions from the involutions $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$, then become,

$$
\begin{aligned}
(d) \quad & 0 = 4ip_z\xi - \left(g_z\sigma^2 - h_z\sigma^1\right)\xi \\
(m) \quad & 0 = \frac{\nu}{2f_4}\sigma^2\xi^* + \frac{D_zf_4}{2f_4}\xi + \frac{1}{2}f_z\sigma^3\xi - \frac{i}{16}\left(g_z\sigma^2 + h_z\sigma^1\right)\xi \\
(i_1) \quad & 0 = -\frac{i}{2f_1}\sigma^3\xi^* + \frac{D_zf_1}{2f_1}\xi - \frac{1}{2}f_z\sigma^3\xi - \frac{i}{16}\left(-3g_z\sigma^2 + h_z\sigma^1\right)\xi \\
(i_2) \quad & 0 = -\frac{\nu}{2f_2}\xi^* + \frac{D_zf_2}{2f_2}\xi - \frac{1}{2}f_z\sigma^3\xi - \frac{i}{16}\left(g_z\sigma^2 - 3h_z\sigma^1\right)\xi \\
(a+) \quad & 0 = D_z\xi^* + \frac{i}{2}\hat{\omega}_z\xi^* + f_z\sigma^3\xi^* + \frac{i}{4}\left(g_z\sigma^2 - h_z\sigma^1\right)\xi^* \\
(a-) \quad & 0 = D_z\xi - \frac{i}{2}\hat{\omega}_z\xi + \frac{i}{8}\left(g_z\sigma^2 + h_z\sigma^1\right)\xi
\end{aligned}
\tag{6.20}
$$

6.6 Normalizations of f_1, f_2, f_4

We shall now extract all information contained in equations (d), (m), (i_1), and (i_2). To do this, for each equation, we form two functionally independent linear combinations; since each equation is a 2-component matrix, these two linear combinations will fully capture the contents of the corresponding equation.

The first linear combinations are obtained by contracting the equations (m), (i_1), and (i_2) on the left respectively by $2\xi^\dagger$, $2\xi^\dagger\sigma^1$, and $\xi^\dagger\sigma^2$. The first term in each resulting equation vanishes by antisymmetry of σ^2, and we obtain,

$$
\begin{aligned}
(m) \quad & 0 = \frac{D_zf_4}{f_4}\xi^\dagger\xi + f_z\xi^\dagger\sigma^3\xi - \frac{i}{8}\xi^\dagger\left(g_z\sigma^2 + h_z\sigma^1\right)\xi \\
(i_1) \quad & 0 = \frac{D_zf_1}{f_1}\xi^\dagger\sigma^1\xi + f_z\xi^\dagger\sigma^3\sigma^1\xi - \frac{i}{8}\xi^\dagger\left(-3ig_z\sigma^3 + h_z\right)\xi \\
(i_2) \quad & 0 = \frac{D_zf_2}{f_2}\xi^\dagger\sigma^2\xi + f_z\xi^\dagger\sigma^3\sigma^2\xi - \frac{i}{8}\xi^\dagger\left(g_z + 3ih_z\sigma^3\right)\xi
\end{aligned}
\tag{6.21}
$$

The following combinations for $p = 0, 1, 2$ are calculated using the equations $(a\pm)$,

$$
D_z\left(\xi^\dagger\sigma^p\xi\right) = -f_z\xi^\dagger\sigma^3\sigma^p\xi - \frac{i}{8}\xi^\dagger\left(\sigma^p(g_z\sigma^2 + h_z\sigma^1) - 2(g_z\sigma^2 + h_z\sigma^1)\sigma^p\right)\xi
\tag{6.22}
$$

They combine with (6.21) to give

$$
D_z\left(f_4^{-1}\xi^\dagger\xi\right) = D_z\left(f_1^{-1}\xi^\dagger\sigma^1\xi\right) = D_z\left(f_2^{-1}\xi^\dagger\sigma^2\xi\right) = 0
\tag{6.23}
$$

so that these ratios are constants. The BPS equations are linear in ξ, ξ^* and invariant under scaling by a real constant, which allows us to normalize the relation involving f_4, as follows,

$$
\xi^\dagger\xi = f_4 \qquad\qquad \xi^\dagger\sigma^i\xi = \lambda_i f_i \qquad\qquad i = 1, 2
\tag{6.24}
$$

where λ_1, λ_2 are real constants.

The second set of linear combinations is obtained by contracting the equations (d), (m), (i_1), and (i_2) on the left respectively by $\xi^t\sigma^2$, so as to eliminate the derivative terms in f_i. We also make use of (6.24) to obtain,

$$
\begin{aligned}
(d) \quad & 0 = g_z\,\xi^t\xi + ih_z\,\xi^t\sigma^3\xi \\
(m) \quad & 0 = i\nu - f_z\xi^t\sigma^1\xi + \frac{1}{8}\xi^t(g_z - ih_z\sigma^3)\xi \\
(i_1) \quad & 0 = -i\lambda_1 - f_z\xi^t\sigma^1\xi - \frac{1}{8}\xi^t(-3g_z - ih_z\sigma^3)\xi \\
(i_2) \quad & 0 = -i\nu\lambda_2 - f_z\xi^t\sigma^1\xi - \frac{1}{8}\xi^t(g_z + 3ih_z\sigma^3)\xi
\end{aligned}
\tag{6.25}
$$

In the sum $2(m) - (i_1) - (i_2)$, all dependence on f_z, g_z, h_z cancels, and using (6.24), we obtain $\nu\lambda_1 + \lambda_2 = -2$. From the sum $(i_1) - (i_2)$, and using (d), we find $\nu\lambda_1 = \lambda_2$, so that $\lambda_1 = -\nu$, and $\lambda_2 = -1$. Putting all together, the final expressions are,

$$
\begin{aligned}
f_4 &= \xi^\dagger\xi & &= \alpha\bar\alpha + \beta\bar\beta \\
f_1 &= -\nu\xi^\dagger\sigma^1\xi & &= -\nu(\alpha\bar\beta + \beta\bar\alpha) \\
f_2 &= -\xi^\dagger\sigma^2\xi & &= i(\beta\bar\alpha - \alpha\bar\beta)
\end{aligned}
\tag{6.26}
$$

These relations completely solve the (i_1) and (i_2) equations, so that only the (d), (m) and $(a\pm)$ equations remain to be solved.

6.7 Consistency

To solve for the reality conditions in subsection 6.3, we had made the assumption that $\zeta^\dagger\tau^{(323)}\sigma^a\zeta = \nu\zeta^\dagger\tau^{(311)}\sigma^a\zeta$ is non-vanishing. In terms of the parametrization of ζ by α and β, this quantity takes on the following form, $\zeta^\dagger\tau^{(323)}\sigma^z\zeta = -2i\alpha\beta$. Generically, this quantity must be non-vanishing on regular solutions, since its vanishing would imply that $f_1 = f_2 = 0$ identically. Thus, our earlier non-vanishing assumption is consistent.

6.8 The reduced BPS equations in conformal coordinates

We choose conformal complex coordinates w on Σ, such that the metric takes the form, $ds_\Sigma^2 = 4\rho^2 d\bar w\, dw$. The frames, derivatives, and connection are then given by

$$
\begin{aligned}
e^z &= \rho\,dw & D_z &= \rho^{-1}\partial_w & \hat\omega_z &= +i\rho^{-2}\partial_w\rho \\
e^{\bar z} &= \rho\,d\bar w & D_{\bar z} &= \rho^{-1}\partial_{\bar w} & \hat\omega_{\bar z} &= -i\rho^{-2}\partial_{\bar w}\rho
\end{aligned}
\tag{6.27}
$$

Notice that z and $\bar z$ are *frame indices*, whence the extra factor of $1/\rho$ in D_z, $D_{\bar z}$.

With the help of the relation (6.18), we may express ξ in terms of $\alpha, \beta, \bar\alpha$ and $\bar\beta$ in the remaining equations. These equations are: the dilatino equation (d), the differential equations $(a+)$ and $(a-)$ of (6.20), and the single equation (m) of (6.25). Expressed in

conformal complex coordinates w, these equations become,

$$(d) \qquad 4p_z\alpha + (g_z - ih_z)\beta = 0$$
$$4p_z\beta - (g_z + ih_z)\alpha = 0$$
$$(m) \qquad i\,\nu - 2\alpha\beta f_z + \frac{1}{8}(g_z - ih_z)\alpha^2 + \frac{1}{8}(g_z + ih_z)\beta^2 = 0$$
$$(a+) \qquad \frac{1}{\rho}\partial_w\bar{\alpha} - \frac{1}{2\rho^2}(\partial_w\rho)\bar{\alpha} + f_z\bar{\alpha} + \frac{1}{4}(g_z - ih_z)\bar{\beta} = 0$$
$$\frac{1}{\rho}\partial_w\bar{\beta} - \frac{1}{2\rho^2}(\partial_w\rho)\bar{\beta} - f_z\bar{\beta} - \frac{1}{4}(g_z + ih_z)\bar{\alpha} = 0$$
$$(a-) \qquad \frac{1}{\rho}\partial_w\alpha + \frac{1}{2\rho^2}(\partial_w\rho)\alpha + \frac{1}{8}(g_z + ih_z)\beta = 0$$
$$\frac{1}{\rho}\partial_w\beta + \frac{1}{2\rho^2}(\partial_w\rho)\beta - \frac{1}{8}(g_z - ih_z)\alpha = 0 \qquad (6.28)$$

This system will be the starting point of our construction of the complete and exact solution.

6.9 Constant dilaton implies $AdS_5 \times S^5$

In this subsection we show that the only solution of the BPS equations (6.28) with constant dilaton is $AdS_5 \times S^5$. The argument is presented independently from the general solution which will be derived in the next section.

A constant dilaton implies $p_z = 0$. It follows from the fact that the metric factors (6.26) cannot be identically zero, and equations (d) and $(a-)$ of (6.28) that the 3-form fluxes have to vanish, i.e. $g_z = h_z = 0$. Solving equation $(a-)$ implies

$$\rho\alpha^2 = \bar{A}(\bar{w})$$
$$\rho\beta^2 = i\bar{B}(\bar{w}) \qquad (6.29)$$

where $A(w)$ and $B(w)$ are purely holomorphic functions of w. The phase is chosen for later convenience. From the difference of the $(a+)$ equations it follows that

$$\partial_w\left(\frac{AB}{\rho^4}\right) = 0 \qquad (6.30)$$

and hence $\rho^4 = |AB|^2$. It is therefore possible to make a holomorphic change of coordinates and set $\rho = 1$. This implies $A = 1/B$. The remaining equation of $(a-)$ leads to an equation for $A(w)$

$$\partial_w A + \nu A = 0 \qquad (6.31)$$

which is solved by $A = e^{-\nu w}$ and hence

$$\alpha = e^{-\nu w/2}$$
$$\beta = ie^{\nu w/2} \qquad (6.32)$$

The metric factors are given by

$$
\begin{aligned}
f_1 &= +2\sin y \\
f_2 &= -2\cos y \\
f_4 &= +2\,\mathrm{ch}x
\end{aligned}
\tag{6.33}
$$

where the coordinates x, y are related to w by $w = x + iy$ and take values in the strip $x \in [-\infty, \infty], y \in [0, \pi]$. Hence the ten-dimensional metric is given by

$$
ds^2 = 4\Big(dx^2 + \mathrm{ch}^2 x \, ds^2_{AdS_4} + dy^2 + \cos^2 y \, ds^2_{S^2} + \sin^2 y \, ds^2_{S^2}\Big)
\tag{6.34}
$$

which is indeed $AdS_5 \times S^5$. Since a constant dilaton implies $AdS_5 \times S^5$, we shall henceforth assume that the dilaton has a non-trivial variation over Σ.

7. The BPS equations form an integrable system

In this section, we shall show that the BPS equations form an integrable system. The solutions to this system will automatically solve the Bianchi identities and field equations of Type IIB supergravity as discussed in section 8. As a first step, we solve the (d) and $(a-)$ part of the system, and then use its solution to reduce further the remaining (m) and $(a+)$ equations to a system of first order equations on two real scalar fields, the dilaton ϕ and the Σ-metric factor ρ. We shall then show that this system is automatically integrable.

First, we shall view the dilatino equations (d) as determining g_z and h_z in terms of α, β and p_z (with corresponding equations for their complex conjugates),

$$
\begin{aligned}
g_z - ih_z &= -\frac{4\alpha}{\beta}\, p_z \\
g_z + ih_z &= +\frac{4\beta}{\alpha}\, p_z
\end{aligned}
\tag{7.1}
$$

and the (m) equation as determining f_z,

$$
f_z = \frac{i\nu}{2\alpha\beta} - \frac{\alpha^4 - \beta^4}{4\alpha^2\beta^2}p_z
\tag{7.2}
$$

These equations may be used to eliminate f_z, g_z and h_z from the remaining equations, $(a+)$ and $(a-)$. We recast the resulting equations in the following form, ready for later use,

$$
(a+) \qquad \partial_w \ln\left(\frac{\bar\alpha}{\bar\beta}\right) - \frac{1}{\rho}\left(\frac{\alpha\bar\beta}{\bar\alpha\beta} - \frac{\bar\alpha\beta}{\alpha\bar\beta}\right) p_z + \frac{i\nu\rho}{\alpha\beta} - \frac{\alpha^4 - \beta^4}{2\alpha^2\beta^2}\rho p_z = 0
$$

$$
\partial_w \ln(\bar\alpha\bar\beta) - \partial_w \ln\rho - \frac{1}{\rho}\left(\frac{\alpha\bar\beta}{\bar\alpha\beta} + \frac{\bar\alpha\beta}{\alpha\bar\beta}\right) p_z = 0
\tag{7.3}
$$

$$
(a-) \qquad \partial_w \alpha + \frac{1}{2\rho}(\partial_w\rho)\alpha + \frac{1}{2}\frac{\beta^2}{\alpha}\rho\, p_z = 0
$$

$$
\partial_w \beta + \frac{1}{2\rho}(\partial_w\rho)\beta + \frac{1}{2}\frac{\alpha^2}{\beta}\rho\, p_z = 0
\tag{7.4}
$$

7.1 Solution to the $(a-)$ system

Multiplying the first equation of (7.4) by $2\rho\alpha$ and the second by $2\rho\beta$, we obtain equivalently,

$$\partial_w\left(\rho\alpha^2\right) + \rho\beta^2\,\rho p_z = 0$$
$$\partial_w\left(\rho\beta^2\right) + \rho\alpha^2\,\rho p_z = 0 \tag{7.5}$$

Adding and subtracting both, we get

$$\partial_w \ln\left(\rho(\alpha^2 + \beta^2)\right) + \rho p_z = 0$$
$$\partial_w \ln\left(\rho(\alpha^2 - \beta^2)\right) - \rho p_z = 0 \tag{7.6}$$

It follows immediately from these equations that ρp_a is the gradient of a scalar function. Inspection of (3.1) and (3.3) reveals that this scalar function is none other than ϕ (related to the dilaton Φ in standard normalization by $\phi = \Phi/2$), so that we deduce from the BPS equations the relation

$$\rho p_z = \partial_w \phi \tag{7.7}$$

The system (7.6) may now be solved completely in terms of two (locally) holomorphic functions $\kappa(w)$ and $\lambda(w)$,

$$\rho(\alpha^2 + \beta^2) = \bar\kappa\, e^{-\bar\lambda - \phi}$$
$$\rho(\alpha^2 - \beta^2) = \bar\kappa\, e^{+\bar\lambda + \phi} \tag{7.8}$$

From the (m) equation, it is manifest that α and β are spinors with respect to the SO(2) frame group of Σ with weight $(-1/2, 0)$ in a convention in which e^z has weight $(-1, 0)$. Since ρ has weight $(1/2, 1/2)$, we conclude that κ is actually a form of weight $(1/2, -1/2) \sim (1, 0)$. These relations may be solved for α and β, as follows,

$$\alpha = \left(\frac{\bar\kappa}{\rho}\right)^{\frac{1}{2}} \mathrm{ch}(\phi + \bar\lambda)^{\frac{1}{2}}$$
$$\beta = i\left(\frac{\bar\kappa}{\rho}\right)^{\frac{1}{2}} \mathrm{sh}(\phi + \bar\lambda)^{\frac{1}{2}} \tag{7.9}$$

Here, we have adopted a definite sign choice for each square root. The parametrization in terms of κ and λ is convenient since one natural combination will involve only ϕ and λ, while another will involve only ρ and κ. They are given by[9]

$$\frac{\alpha^2}{\beta^2} = \frac{1 + e^{2\phi + 2\bar\lambda}}{1 - e^{2\phi + 2\bar\lambda}}$$
$$\bar\kappa^2 = \rho^2\left(\alpha^4 - \beta^4\right) \tag{7.10}$$

Another equation, which may be directly deduced from (7.9), will be useful as well,

$$4\rho^2\alpha^2\beta^2 = \bar\kappa^2\left(e^{-2\phi - 2\bar\lambda} - e^{2\phi + 2\bar\lambda}\right) \tag{7.11}$$

---

[9]The corresponding equation (5.16) of [31] is incorrect.

7.2 Solution to the $(a+)$ system

The $(a+)$ system will be solved as follows. Equations (7.9) will be viewed as giving α and β in terms of ϕ, ρ and the holomorphic functions κ and λ, and will be used to eliminate $\bar{\alpha}$ and $\bar{\beta}$ from the $(a+)$ system. The combinations of ∂_w-derivatives of logarithms occurring in (7.3) may be computed from the solution of the $(a-)$ equations, by taking the derivatives of the complex conjugates to the first relation in (7.10) and the equation of (7.11),

$$\partial_w \ln\left(\frac{\bar{\alpha}}{\bar{\beta}}\right) = \frac{1}{2}\left(\frac{\bar{\alpha}^2}{\bar{\beta}^2} - \frac{\bar{\beta}^2}{\bar{\alpha}^2}\right)(\partial_w \phi + \partial_w \lambda)$$

$$\partial_w \ln\left(\rho\bar{\alpha}\bar{\beta}\right) = \partial_w \ln\kappa - \frac{1}{2}\left(\frac{\bar{\alpha}^2}{\bar{\beta}^2} + \frac{\bar{\beta}^2}{\bar{\alpha}^2}\right)(\partial_w \phi + \partial_w \lambda) \tag{7.12}$$

Eliminating now from both $(a+)$ equations in (7.3), p_z using (7.7), the w-derivatives of $\ln\bar{\alpha}$ and $\ln\bar{\beta}$ in (7.3) using (7.12), and the remaining algebraic dependence of α, β and their complex conjugates, using (7.8)–(7.11), we obtain a system of first order differential equations for ϕ and ρ only, with κ and λ viewed as given holomorphic functions,

$$\left[\frac{1}{\text{sh}(2\phi+2\lambda)} - \frac{1}{\text{sh}(2\phi+2\bar{\lambda})} - \frac{2\,\text{sh}(\lambda-\bar{\lambda})}{|\text{sh}(2\phi+2\lambda)|}\right]\partial_w\phi = \frac{\sqrt{2}\nu\,\bar{\kappa}^{-1}\rho^2}{\text{sh}(2\phi+2\bar{\lambda})^{\frac{1}{2}}} - \frac{\partial_w\lambda}{\text{sh}(2\phi+2\lambda)} \tag{7.13}$$

$$\frac{2}{\rho}\partial_w\rho = \partial_w\ln\kappa + \frac{\text{ch}(2\phi+2\lambda)}{\text{sh}(2\phi+2\lambda)}(\partial_w\phi+\partial_w\lambda) + 2\frac{\text{sh}(2\phi+\lambda+\bar{\lambda})}{|\text{sh}(2\phi+2\lambda)|}\partial_w\phi \tag{7.14}$$

One can now use the first equation to eliminate $\partial_w\phi$ from the second equation, so that we get two first order equations separately for $\partial_w\phi$ and $\partial_w\ln\rho$. Each of these equations has a complex conjugate giving the $\partial_{\bar{w}}$-derivatives, and there will be two integrability conditions.

7.3 Relation to a new integrable system

To begin, we recast (7.14) in terms of its complex conjugate equation, rearranged as follows,

$$\partial_{\bar{w}}\ln\left(\frac{\sqrt{2}\nu\rho^2}{\bar{\kappa}\,\text{sh}(2\phi+2\bar{\lambda})^{\frac{1}{2}}}\right) = 2\frac{\text{sh}(2\phi+\lambda+\bar{\lambda})}{|\text{sh}(2\phi+2\lambda)|}\partial_{\bar{w}}\phi \tag{7.15}$$

From this form of (7.14), we see that ρ and κ may be eliminated between (7.13) and (7.14), by eliminating the combination $\sqrt{2}\nu\rho^2\bar{\kappa}^{-1}\text{sh}(2\phi+2\bar{\lambda})^{-\frac{1}{2}}$ between (7.13) and (7.15). The resulting equation involves only ϕ and λ, though this "simplification" has been achieved at the cost of obtaining a second order partial differential equation, given by

$$\partial_w\partial_{\bar{w}}\phi - 2\frac{\text{sh}(4\phi+2\lambda+2\bar{\lambda})}{|\text{sh}(2\phi+2\lambda)|^2}\partial_w\phi\,\partial_{\bar{w}}\phi \tag{7.16}$$

$$= \frac{\partial_{\bar{w}}\phi\,\partial_w\lambda}{\text{ch}(\lambda-\bar{\lambda})}\frac{\text{sh}(2\phi+2\bar{\lambda})}{\text{sh}(2\phi+2\lambda)}\left(-\frac{1}{\text{sh}(\lambda-\bar{\lambda})} - \frac{\text{sh}(2\phi+2\lambda)^{\frac{1}{2}}}{\text{sh}(2\phi+2\bar{\lambda})^{\frac{1}{2}}}\right) + \text{c.c.}$$

Note that (7.16) is real. Although the equation looks daunting, we shall show that it is integrable and better even, that its general solution may be obtained in analytic form.

7.4 Integrability

In this subsection, we shall show that the system of first order differential equations (7.13), (7.14) and their complex conjugates, form an integrable system for any choice of holomorphic functions κ and λ. Equation (7.16), which was shown to be a consequence of the first order system (7.13) and (7.14), will be used in the process.

Integrability in ρ amounts to the reality of $\partial_{\bar{w}}\partial_w \ln \rho^2$. This quantity may be obtained directly by taking the $\partial_{\bar{w}}$-derivative of (7.14). By construction, the resulting integrability equation does not involve ρ or κ, and actually coincides with the second order equation (7.16), which we have already show to be a consequence of the system (7.13) and (7.14). Thus, integrability in ρ holds automatically for any holomorphic κ and λ.

Integrability in ϕ may be verified by taking the $\partial_{\bar{w}}$ derivative of (7.13). As was already established in subsection 7.3, the resulting equation, after elimination of ρ and κ is precisely the second order equation (7.16). The fact that (7.16) emerges as a real equation directly guarantees that $\partial_{\bar{w}}\partial_w \phi$ is real, and thus that the system (7.13) and (7.14) is integrable in ϕ.

8. Reduced Bianchi identities and field equations

In this section, we shall derive the reduced Bianchi identities, and field equations, and show that they are satisfied for any field configuration that satisfies the BPS equations for 16 supersymmetries. As shown in subsection 6.4, every solution of the BPS equations may be transformed to a solution with vanishing axion field under an $SU(1,1)$ transformation. Thus, we may restrict to the case of vanishing axion, without loss of generality. The solutions with non-vanishing axion may be obtained by making $SU(1,1)$ transformations.

8.1 The reduced Bianchi identities

Using differential form notation, it is straightforward to reduce the Bianchi identities on the Ansatz defined in section 3. We find,

$$
\begin{aligned}
0 &= dP - 2iQ \wedge P \\
0 &= dQ + iP \wedge \bar{P} \\
0 &= d\mathcal{G} + 2(d\ln f_1) \wedge \mathcal{G} - iQ \wedge \mathcal{G} + P \wedge \bar{\mathcal{G}} \\
0 &= d\mathcal{H} + 2(d\ln f_2) \wedge \mathcal{H} - iQ \wedge \mathcal{H} - P \wedge \bar{\mathcal{H}} \\
0 &= d\mathcal{F} + 4(d\ln f_4) \wedge \mathcal{F} \\
0 &= d(*_2\mathcal{F}) + 2\left(d\ln(f_1 f_2)\right) \wedge *_2\mathcal{F} - \frac{1}{8}\left(\mathcal{G} \wedge \bar{\mathcal{H}} + \bar{\mathcal{G}} \wedge \mathcal{H}\right)
\end{aligned}
\tag{8.1}
$$

The reduced Bianchi identities now simplify as one may set $Q = 0$ in (8.1).

8.2 Derivation of the reduced field equations

In this subsection we reduce the Type IIB field equations of (3.9), (3.10), and (3.11) to the two-parameter Ansatz of section 4 for $Q = 0$.

8.2.1 The dilaton field equation

It is straightforward to derive the dilaton equation, using the convention $f^2 dB = d\phi$,

$$D^a D_a \phi + 2(D^a \phi) D_a \ln(f_1 f_2 f_4) + \frac{1}{4}(g_a g^a - h_a h^a) = 0 \tag{8.2}$$

In local conformal coordinates, this becomes,

$$\partial_{\bar{w}} \partial_w \phi + (\partial_{\bar{w}} \phi) \partial_w \ln(f_1 f_2 f_4^2) + (\partial_w \phi) \partial_{\bar{w}} \ln(f_1 f_2 f_4^2) + \frac{1}{4}\rho^2 (g_z g_{\bar{z}} - h_z h_{\bar{z}}) = 0 \tag{8.3}$$

8.2.2 The G-field equation

To reduce the field equations of the antisymmetric tensor field $B_{(2)}$, it is convenient to first recast (3.10) in terms of differential forms,[10]

$$*d(*G) + (i_P \bar{G}) - 4i\,(i_G F_{(5)}) = 0 \tag{8.4}$$

Here $i_V G$ stands for the contraction of G with V. Some useful intermediary results for this calculation are as follows,

$$i_P \bar{G} = p^a g_a e^{45} - i p^a h_a e^{67}$$
$$i_G F_{(5)} = f_a \varepsilon^{ab} g_b e^{67} + i f_a \varepsilon^{ab} h_b e^{45} \tag{8.5}$$

The G-field equation then reduces to the following two real equations,

$$D^a g_a + 2g_a D^a \ln(f_2 f_4^2) - p^a g_a - 4 f_a \varepsilon^{ab} h_b = 0$$
$$D^a h_a + 2h_a D^a \ln(f_1 f_4^2) + p^a h_a + 4 f_a \varepsilon^{ab} g_b = 0 \tag{8.6}$$

In conformal gauge, and after multiplication by ρ^2, this simplifies to

$$\partial_w(\rho g_{\bar{z}}) + 2\rho g_{\bar{z}} \partial_w \ln(f_2 f_4^2) - (\partial_w \phi)\rho g_{\bar{z}} + 4i\rho f_z \,\rho h_{\bar{z}} + c.c. = 0$$
$$\partial_w(\rho h_{\bar{z}}) + 2\rho h_{\bar{z}} \partial_w \ln(f_1 f_4^2) + (\partial_w \phi)\rho h_{\bar{z}} - 4i\rho f_z \,\rho g_{\bar{z}} + c.c. = 0 \tag{8.7}$$

8.2.3 Einstein's equations

The Einstein equations, respectively for the components mn, $i_1 j_1$, $i_2 j_2$, and ab, are as follows, (all other components must vanish by $SO(2,3) \times SO(3) \times SO(3)$ symmetry),

$$0 = -\frac{3}{f_4^2} - 3\frac{|D_a f_4|^2}{f_4^2} - 2\frac{D^a f_4 D_a(f_1 f_2)}{f_1 f_2 f_4} - \frac{D^a D_a f_4}{f_4} + 4 f_a f^a + \frac{1}{8} g_a g^a + \frac{1}{8} h_a h^a$$

$$0 = \frac{1}{f_1^2} - \frac{|D_a f_1|^2}{f_1^2} - 4\frac{D^a f_4 D_a f_1}{f_1 f_4} - 2\frac{D^a f_1 D_a f_2}{f_1 f_2} - \frac{D^a D_a f_1}{f_1} - 4 f_a f^a - \frac{3}{8} g_a g^a + \frac{1}{8} h_a h^a$$

$$0 = \frac{1}{f_2^2} - \frac{|D_a f_2|^2}{f_2^2} - 4\frac{D^a f_4 D_a f_2}{f_2 f_4} - 2\frac{D^a f_1 D_a f_2}{f_1 f_2} - \frac{D^a D_a f_2}{f_2} - 4 f_a f^a + \frac{1}{8} g_a g^a - \frac{3}{8} h_a h^a$$

$$0 = -2\frac{D_b D_a f_4}{f_4} - \frac{D_b D_a f_1}{f_1} - \frac{D_b D_a f_2}{f_2} + \frac{1}{2} R^{(2)} \delta_{ab} - D_a \phi D_b \phi$$
$$\qquad -2\delta_{ab} f_c f^c + 4 f_a f_b + \frac{1}{16} \delta_{ab}(g_c g^c + h_c h^c) - \frac{1}{4} g_a g_b - \frac{1}{4} h_a h_b \tag{8.8}$$

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[10]Our conventions for the Poincaré dual are given via the following pairing relation between two arbitrary rank p differential forms $S_{(p)}$ and $T_{(p)}$, by $S_{(p)} \wedge *T_{(p)} = \frac{1}{p!} S_{(p)}^{a_1 \cdots a_p} T_{(p)a_1 \cdots a_p} e^{0123456789}$. In particular, we have $**S_{(p)} = (-1)^{p+1} S_{(p)}$, and the duals $*e^{01234} = -e^{56789}$, and $*e^{6789} = e^{012345}$, $*e^{45a} = +\varepsilon^{ab} e^{012367b}$, $*e^{01234589} = -e^{67}$, $*e^{67a} = +\varepsilon^{ab} e^{012345b}$, and $*e^{01236789} = -e^{45}$, which are useful in deriving the G-equations.

With respect to the frame rotation group SO(2) of Σ, the first three equations are weight $(0,0)$, while the last contains both weights $(0,0)$, and $(2,0)$. It will be useful to separate these two parts in the last equation. The weight $(2,0)$ part is given by

$$2\frac{D_z^2 f_4}{f_4} + \frac{D_z^2 f_1}{f_1} + \frac{D_z^2 f_2}{f_2} + (D_z\phi)^2 - 4f_z^2 + \frac{1}{4}g_z^2 + \frac{1}{4}h_z^2 = 0 \tag{8.9}$$

and its complex conjugate, It may be viewed as the *constraint* of vanishing spin 2 parts of the stress tensor on Σ. The spin 0 part is given by

$$-2\frac{D^a D_a f_4}{f_4} - \frac{D^a D_a f_1}{f_1} - \frac{D^a D_a f_2}{f_2} + R^{(2)} - D_a\phi D^a\phi - \frac{1}{8}\left(g_c g^c + h_c h^c\right) = 0 \tag{8.10}$$

8.3 Derivation of Bianchi identities and field equations from BPS equations

The Bianchi identity for the dilaton of (8.1), namely $dP = 0$, was already derived from the BPS equations in (7.7). The derivation of the remaining Bianchi identities and field equations from the BPS equations is considerably more involved. Below, we shall present the analytic derivations of the dilaton field equation (8.3) and the spin 2 constraint (8.9) of the Einstein equations directly from the BPS equations. The remaining Bianchi identities and field equations were verified to follow from the BPS equations using Mathematica. General arguments that the BPS equations imply the field equations are given in [41].

8.3.1 The dilaton equation

To work out the dilaton equation (8.3), we first derive the following quantities,

$$\rho^2(g_z g_{\bar{z}} - h_z h_{\bar{z}}) = 16\frac{\text{sh}(2\phi + \lambda + \bar{\lambda})}{|\text{sh}(2\phi + 2\lambda)|}\,\partial_w\phi\,\partial_{\bar{w}}\phi$$
$$\partial_w \ln(f_1 f_2 f_4^2) = -2\partial_w \ln\rho^2 + 2\partial_w\kappa + \partial_w \ln\text{sh}(\lambda - \bar{\lambda})$$
$$+\partial_w \ln\left\{\text{ch}(2\phi + \lambda + \bar{\lambda}) + |\text{sh}(2\phi + 2\lambda)|\right\} \tag{8.11}$$

The first equation is obtained from (7.1) by eliminating α and β using (7.8)–(7.11). The second equation is obtained starting from (6.26) to express f_1, f_2, f_4 in terms of $\alpha, \beta, \bar{\alpha}, \bar{\beta}$, and then using (7.3) and (7.4) to compute the derivatives of $\alpha, \beta, \bar{\alpha}, \bar{\beta}$, after which all algebraic dependence on these variables is eliminated in favor of $\phi, \rho, \kappa, \lambda$ using (7.8)–(7.11). Using now (7.14) to eliminate ρ and κ from the above expression, we obtain,

$$\partial_w \ln(f_1 f_2 f_4^2) = -\partial_w \ln\text{sh}(2\phi + 2\lambda) - 4\frac{\text{sh}(2\phi + \lambda + \bar{\lambda})}{|\text{sh}(2\phi + 2\lambda)|}\partial_w\phi + \partial_w \ln\text{sh}(\lambda - \bar{\lambda})$$
$$+\partial_w \ln\left\{\text{ch}(2\phi + \lambda + \bar{\lambda}) + |\text{sh}(2\phi + 2\lambda)|\right\} \tag{8.12}$$

Substituting these expressions back into the dilaton equation (8.3), it is clear that we obtain a second order partial differential equation that involves only ϕ and λ. Not surprisingly, this equation coincides with (7.16), which in turn was already shown to follow from the system of first order equations (7.13) and (7.14). Thus, the dilaton equation will be satisfied as soon as the system of equations (7.13) and (7.14) is satisfied.

8.3.2 The constraint equation

The weight $(2,0)$ constraint equation (8.9) is also a consequence of the BPS equations. To see this, we need to compute second order derivatives, using the formula,

$$\rho^2 D_z^2 f_i = \partial_w^2 f_i - (\partial_w \ln \rho^2) \partial_w f_i \qquad\qquad i = 1, 2, 4 \qquad\qquad (8.13)$$

We first derive formulas for the first derivatives of the f_i functions, using the reduced BPS equations of (7.1), (7.2), and (7.3), and find,

$$\partial_w f_1 = (+\partial_w \phi - A_+) f_1 + \frac{\rho}{2\alpha\beta} f_2$$

$$\partial_w f_2 = (-\partial_w \phi - A_+) f_2 - \frac{\rho}{2\alpha\beta} f_1$$

$$\partial_w f_4 = +A_+ f_4 - \frac{i\nu\rho}{2\alpha\beta} \sqrt{f_4^2 - f_1^2 - f_2^2} \qquad\qquad (8.14)$$

where we use the following objects,

$$A_+ \equiv -\frac{1}{4} \partial_w \ln \left(\mathrm{sh}(2\phi + 2\bar{\lambda}) \right) = \frac{1}{2} \partial_w \ln(\rho\alpha\beta)$$

$$A_- \equiv -\frac{1}{4} \partial_w \ln \left(\frac{\mathrm{sh}(\phi + \bar{\lambda})}{\mathrm{ch}(\phi + \bar{\lambda})} \right) = -\rho f_z + \frac{i\nu\rho}{2\alpha\beta} \qquad\qquad (8.15)$$

Furthermore, the second derivatives may be computed with the help of the reduced BPS equations (7.1), (7.2), and (8.14) and (7.3). After considerable simplifications, we find,

$$4\rho^2 \frac{D_z^2 f_4}{f_4} + 2\rho^2 \frac{D_z^2 f_1}{f_1} + 2\rho^2 \frac{D_z^2 f_2}{f_2} = 4(\partial_w \phi)^2 + 8A_+^2 - 2\frac{\rho^2}{\alpha^2\beta^2} - 8i\frac{\nu\rho}{\alpha\beta} A_-$$

We shall use also the following equations, derived from (7.1) and (7.2),

$$-\frac{1}{2}\rho^2 \left(g_z^2 + h_z^2 \right) = 8(\partial_w \phi)^2$$

$$-8\rho^2 f_z^2 = -8A_-^2 + 8\frac{i\nu\rho}{\alpha\beta} A_- + 2\frac{\rho^2}{\alpha^2\beta^2} \qquad\qquad (8.16)$$

Putting all together, and using $8A_+^2 - 8A_-^2 = 2(\partial_w \phi)^2$, the constraint (8.9) is found to be satisfied, and thus (8.9) follows from the BPS equations.

9. Complete analytic solution

The system of first order equations of (7.13) and (7.14) for the unknown real scalar functions ϕ and ρ appears formidable. Nonetheless, we shall succeed in constructing a sequence of local changes of variables by which this system is exactly mapped into a system of linear equations. These linear equations will be solved exactly in terms of the 2 holomorphic functions κ and λ, appearing already in (7.13) and (7.14).

The task of finding simultaneous changes of variables for ϕ and ρ which simplify the first order equations is made easier by first searching for a helpful change of variables for ϕ only. This is possible, because we have already shown that the first order system (7.13) and (7.14) implies a single second order partial differential equation, (7.16), which involves only ϕ, but not ρ. It is in this equation that we shall identify a first change of variables, just for ϕ.

9.1 A new field for the dilaton ϕ

The key complication in (7.16) is the appearance of a square root of a ratio of hyperbolic functions on the right hand side of (7.16). To uniformize this square root, we define the new real field ϑ by

$$e^{2i\vartheta} \equiv \frac{\text{sh}(2\phi + 2\lambda)}{\text{sh}(2\phi + 2\bar{\lambda})} \tag{9.1}$$

In terms of ϑ, equation (7.16) simplifies considerably, and becomes,

$$\partial_{\bar{w}}\partial_w \vartheta + \frac{1}{\cos\mu}\left(e^{-i\vartheta}\,\partial_{\bar{w}}\vartheta\,\partial_w\lambda + e^{i\vartheta}\,\partial_w\vartheta\,\partial_{\bar{w}}\bar{\lambda}\right) - 2\frac{\sin\mu}{\cos^2\mu}\partial_w\lambda\,\partial_{\bar{w}}\bar{\lambda}\,\cos\vartheta = 0 \tag{9.2}$$

This equation is of the Liouville or sine-Gordon type [32, 33]. Alternatively, it may be recast in the form of a current conservation equation,

$$\partial_{\bar{w}}\left(\partial_w\vartheta - 2\frac{\partial_w\mu}{\cos\mu}e^{-i\vartheta}\right) + \partial_w\left(\partial_{\bar{w}}\vartheta - 2\frac{\partial_{\bar{w}}\mu}{\cos\mu}e^{+i\vartheta}\right) = 0 \tag{9.3}$$

where we use the notation $\lambda - \bar{\lambda} = i\mu$, with μ real and harmonic. Intermediate steps in this calculation are considerably simplified with the help of the following equation,

$$|\text{sh}(2\phi + 2\lambda)|^2 = \frac{(\sin 2\mu)^2}{4\sin(\vartheta + \mu)\sin(\vartheta - \mu)} \tag{9.4}$$

and the derivatives

$$\partial_w\phi = -\frac{\sin 2\mu\,\partial_w\vartheta}{4\sin(\vartheta + \mu)\sin(\vartheta - \mu)} - \frac{i}{2}\partial_w\mu + \frac{\sin 2\vartheta\,\partial_w\mu}{4\sin(\vartheta + \mu)\sin(\vartheta - \mu)} \tag{9.5}$$

9.2 A new field for the metric ρ

Having identified a change of variables for the dilaton ϕ that significantly simplifies (7.16), we shall carry out the same change of variables for the dilaton in the first order system (7.13), (7.14) as well, leaving the metric function ρ unchanged for the time being. Equation (7.13) becomes,

$$\partial_w\vartheta - \left(e^{-i\vartheta} + i\sin\mu\right)\frac{\partial_w\mu}{\cos\mu} = \frac{i\nu\sqrt{2}\rho^2}{\bar{\kappa}}\left(\frac{\sin\mu - \sin\vartheta}{\cos\mu\sin\mu}\right)\text{sh}(2\phi + 2\lambda)^{\frac{1}{2}} \tag{9.6}$$

where we have left the factor $\text{sh}(2\phi + 2\lambda)^{\frac{1}{2}}$ on the right hand side unconverted; it will combine with other factors later. Equation (7.14) becomes,

$$\begin{aligned}
\partial_w \ln\rho^2 &= \left(\frac{i}{2} - \frac{\sin 2\vartheta + 4\sin\mu\,\cos\vartheta}{4\sin(\vartheta + \mu)\sin(\vartheta - \mu)}\right)\partial_w\vartheta \\
&+ \left(\frac{\cos 2\vartheta - \cos 2\mu}{2\sin 2\mu} - i\frac{\sin 2\vartheta}{\sin 2\mu} - i\frac{\cos\vartheta}{\cos\mu}\right)\partial_w\mu \\
&+ \frac{\sin 2\vartheta}{\sin(\vartheta + \mu)\sin(\vartheta - \mu)}\left\{\frac{\cos\vartheta}{2\cos\mu} + \frac{\sin 2\vartheta}{4\sin 2\mu}\right\}\partial_w\mu
\end{aligned} \tag{9.7}$$

Equation (9.7), though apparently complicated, provides a clue as to how ρ^2 should be redefined. The strategy will be to multiply ρ^2 by a factor which absorbs *all* the terms proportional to $\partial_w \vartheta$ on the right hand side of (9.7), except for the term $\frac{i}{2} \partial_w \vartheta$. The change of variables that effects this is given by

$$\rho^8 = \frac{\hat{\rho}^8}{16} \kappa^4 \bar{\kappa}^4 (\sin 2\mu)^2 \frac{\sin \vartheta + \sin \mu}{(\sin \vartheta - \sin \mu)^3} \tag{9.8}$$

The factor $(\sin 2\mu)^{-\frac{1}{2}}$ has been included for later convenience.

9.3 The first order system in terms of the new fields

In terms of ϑ and $\hat{\rho}$, the system of first order equations (7.13) and (7.14) or equivalently, equations (9.6) and (9.7) now simplifies considerably and becomes,

$$\partial_w \vartheta - \left(e^{-i\vartheta} + i \sin \mu \right) \frac{\partial_w \mu}{\cos \mu} = -i\nu \hat{\rho}^2 \kappa\, e^{+i\vartheta/2}$$
$$\partial_w \vartheta - 2e^{-i\vartheta} \frac{\partial_w \mu}{\cos \mu} = -2i\, \partial_w \ln \hat{\rho}^2 \tag{9.9}$$

An alternative way of writing the second equation is obtained by eliminating $\partial_w \vartheta$ between both equations,

$$\partial_w \vartheta = \left(e^{-i\vartheta} + i \sin \mu \right) \frac{\partial_w \mu}{\cos \mu} - i\nu \hat{\rho}^2 \kappa\, e^{+i\vartheta/2}$$
$$\partial_w \ln \hat{\rho}^2 = -\frac{i}{2} \frac{e^{-i\vartheta}}{\cos \mu} \partial_w \mu + \frac{1}{2} \partial_w \ln \cos \mu + \frac{\nu}{2} \hat{\rho}^2 \kappa\, e^{+i\vartheta/2} \tag{9.10}$$

It is readily checked that these two equations, together with their complex conjugate equations, forms a system of equations that is integrable in both the real functions ϑ and $\ln \hat{\rho}^2$.

9.4 The first order system in terms of a single complex field

By taking the sum of the second equation in (9.10) with $i/2$ times the first equation, we eliminate the $e^{-i\vartheta} \partial_w \mu / \cos \mu$ term from both equations and we are left with

$$\partial_w \ln \left(\hat{\rho}^2 e^{i\vartheta/2} \right) = \partial_w \ln \cos \mu + \nu \hat{\rho}^2 \kappa\, e^{+i\vartheta/2} \tag{9.11}$$

Thus, the natural variable is the complex combination $\hat{\rho}^2 e^{i\vartheta/2}$, for which (9.11) gives a first integral. Actually, a slightly more convenient combination is the following,

$$\psi \equiv \frac{\cos \mu}{\hat{\rho}^2} e^{-i\vartheta/2} \tag{9.12}$$

In terms of this new variable, the system (9.10) becomes,

$$\partial_w \psi = -\nu \kappa \cos \mu$$
$$\partial_w \bar{\psi} = i\psi \frac{\partial_w \mu}{\cos \mu} - \bar{\psi} \frac{\sin \mu}{\cos \mu} \partial_w \mu \tag{9.13}$$

Thus, the change of variables from $(\phi, \tilde{\rho}^2)$ to $(\psi, \bar{\psi})$ maps the original first order system into a system of linear equations.

9.5 Integration of the first order system

The system is actually even better than linear, since its first equation in (9.13) may be integrated by quadrature alone. To see this, write all components in terms of the (locally) holomorphic functions κ, λ,

$$\partial_w \psi = -\frac{\nu}{2} \kappa \left(e^{+\lambda - \bar{\lambda}} + e^{-\lambda + \bar{\lambda}} \right) \tag{9.14}$$

We introduce the following (locally) holomorphic scalar functions $\mathcal{A}(w)$ and $\mathcal{B}(w)$, as primitives of the (locally) holomorphic exponentials,[11]

$$\partial_w \mathcal{A}(w) = -\frac{\nu}{2} \kappa(w) e^{+\lambda(w)}$$
$$\partial_w \mathcal{B}(w) = -\frac{\nu}{2} \kappa(w) e^{-\lambda(w)} \tag{9.15}$$

In terms of these functions, the general solution of (9.14) is readily written down explicitly,

$$\psi(w, \bar{w}) = e^{-\bar{\lambda}}(\bar{w}) \mathcal{A}(w) + e^{+\bar{\lambda}}(\bar{w}) \mathcal{B}(w) + \overline{\varphi(w)}$$
$$\bar{\psi}(w, \bar{w}) = e^{-\lambda}(w) \overline{\mathcal{A}(w)} + e^{+\lambda}(w) \overline{\mathcal{B}(w)} + \varphi(w) \tag{9.16}$$

where $\varphi(w)$ is a holomorphic function which remains yet to be determined by the second equation in (9.13). Substituting (9.16) into the second equation in (9.13), we obtain the following equation for φ,

$$\left(e^{\lambda - \bar{\lambda}} + e^{-\lambda + \bar{\lambda}} \right) \partial_w \varphi - \varphi \left(e^{\lambda - \bar{\lambda}} - e^{-\lambda + \bar{\lambda}} \right) \partial_w \lambda - 2\bar{\varphi} \, \partial_w \lambda$$
$$= 2e^{-\bar{\lambda}}(\mathcal{A} + \bar{\mathcal{A}})\partial_w \lambda + 2e^{+\bar{\lambda}}(\mathcal{B} - \bar{\mathcal{B}})\partial_w \lambda \tag{9.17}$$

The inhomogeneous solution is readily identified as $-e^{-\lambda}\mathcal{A} + e^{\lambda}\mathcal{B}$, so we redefine φ in terms of a new holomorphic function φ_0, by

$$\varphi \equiv \varphi_0 - e^{-\lambda}\mathcal{A} + e^{\lambda}\mathcal{B}$$
$$\bar{\varphi} \equiv \bar{\varphi}_0 - e^{-\bar{\lambda}}\bar{\mathcal{A}} + e^{\bar{\lambda}}\bar{\mathcal{B}} \tag{9.18}$$

where φ_0 now satisfies the homogeneous equation,

$$\left(e^{\lambda - \bar{\lambda}} + e^{-\lambda + \bar{\lambda}} \right) \partial_w \varphi_0 - \varphi_0 \left(e^{\lambda - \bar{\lambda}} - e^{-\lambda + \bar{\lambda}} \right) \partial_w \lambda - 2\bar{\varphi}_0 \, \partial_w \lambda = 0 \tag{9.19}$$

To solve this equation, we set $\varphi_0 = e^{\lambda}\varphi_1$, where φ_1 is again holomorphic, and satisfies the equation,

$$2(\varphi_1 - \bar{\varphi}_1)\partial_w \lambda + \left(1 + e^{2\lambda - 2\bar{\lambda}} \right) \partial_w \varphi_1 = 0 \tag{9.20}$$

Taking the $\partial_{\bar{w}}$ derivative of the entire equation, and rearranging factors, we get

$$\frac{e^{2\lambda}\partial_w \varphi_1}{\partial_w \lambda} = -\frac{e^{2\bar{\lambda}}\partial_{\bar{w}}\bar{\varphi}_1}{\partial_{\bar{w}}\bar{\lambda}} \tag{9.21}$$

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[11]Notice that the sign ν merely changes the sign of both $\mathcal{A}$ and $\mathcal{B}$.

The left hand side is holomorphic, while the right hand side is anti-holomorphic. The above equality then requires that both ratios be constant and purely imaginary, a number we shall denote by $2ir_1$ with r_1 real. The remaining equation is then

$$\partial_w \varphi_1 = 2ir_1 e^{-2\lambda} \partial_w \lambda \tag{9.22}$$

whose general solution is given by $\varphi_1 = -ir_1 e^{-2\lambda} + r_2$, with $\partial_w r_2 = 0$. Since φ_1 is holomorphic, r_2 must be a complex constant. Substituting this result into the full equation (9.20) for φ_1, we find the additional requirement that r_2 must be real. The most general solution for φ_0 is thus, $\varphi_0 = -ir_1 e^{-\lambda} + r_2 e^{\lambda}$ with r_1, r_2 real. The constants r_1 and r_2 can be absorbed into the functions $\mathcal{A}$ and $\mathcal{B}$, so that the most general solution for ψ is given by

$$\psi = e^{-\bar{\lambda}}(\mathcal{A} - \bar{\mathcal{A}}) + e^{+\bar{\lambda}}(\mathcal{B} + \bar{\mathcal{B}}) \tag{9.23}$$

It will often be convenient to express the results directly in terms of two real harmonic functions h_1 and h_2, instead of the holomorphic functions $\mathcal{A}$ and $\mathcal{B}$. The relation between the two sets of functions is as follows,

$$\mathcal{A} - \bar{\mathcal{A}} = ih_1 \qquad\qquad \psi = +ih_1 e^{-\bar{\lambda}} + h_2 e^{\bar{\lambda}}$$
$$\mathcal{B} + \bar{\mathcal{B}} = h_2 \qquad\qquad \bar{\psi} = -ih_1 e^{-\lambda} + h_2 e^{\lambda} \tag{9.24}$$

In general, these harmonic functions are independent of one another just as the holomorphic functions κ and λ were independent.

9.6 Explicit solution for the dilaton

The dilaton is given in terms of the variable ϑ, which in turn is given by

$$e^{i\vartheta} = \frac{\bar{\psi}}{\psi} = \frac{-ie^{-\lambda}h_1 + e^{\lambda}h_2}{+ie^{-\bar{\lambda}}h_1 + e^{\bar{\lambda}}h_2} \tag{9.25}$$

The dilaton field ϕ is related to ϑ via (9.1), or equivalently, via

$$e^{4\phi + 2\lambda + 2\bar{\lambda}} = \frac{\operatorname{sh}(i\vartheta + \lambda - \bar{\lambda})}{\operatorname{sh}(i\vartheta - \lambda + \bar{\lambda})} \tag{9.26}$$

From $\mathcal{A}$ and $\mathcal{B}$ in (9.15), we derive formulas for κ and $e^{2\lambda}$ in terms of h_1 and h_2,

$$\kappa^2 = 4i \partial_w h_1 \partial_w h_2 \qquad\qquad e^{2\lambda} = i \frac{\partial_w h_1}{\partial_w h_2} \tag{9.27}$$

The dilaton solution may be expressed solely in terms of the harmonic functions h_1 and h_2,

$$e^{4\phi} = \frac{2h_1 h_2 |\partial_w h_2|^2 - h_2^2 W}{2h_1 h_2 |\partial_w h_1|^2 - h_1^2 W} \tag{9.28}$$

where the following combination W will occur ubiquitously,

$$W \equiv \partial_w h_1 \partial_{\bar{w}} h_2 + \partial_w h_2 \partial_{\bar{w}} h_1 \tag{9.29}$$

This then gives the result for the dilaton, announced in (1.2) of the Introduction. Note that it is possible for the right hand side of (9.28) to be negative. Therefore, in order for it to be a valid solution of the BPS equations (7.13), (7.14), as well as Type IIB supergravity, there is an additional restriction on the harmonic functions h_1 and h_2. They must be chosen so that (9.28) is positive on the right hand side. Construction of such harmonic functions is non-trivial. The Janus solution, will be given in section 10, while an infinite class of such harmonic functions is constructed in a companion paper [1].

9.7 Explicit solution for the metric factors

The metric factor ρ^2 is readily calculated by taking the norm of ψ in (9.12), and using the conversion formula (9.8), and is found to be,

$$\rho^8 = \frac{W^2}{h_1^3 h_2^3}\left(2h_1|\partial_w h_2|^2 - h_2 W\right)\left(2h_2|\partial_w h_1|^2 - h_1 W\right) \tag{9.30}$$

The metric factors f_1, f_2, and f_4 are given in terms of the spinor variables α and β by (6.26), which in turn are given via (7.9) in terms of ϕ and κ and λ. The latter are obtained in terms of the harmonic functions h_1 and h_2 using (9.15). It will sometimes be useful to keep the dilaton ϕ and Σ-metric ρ^2, as their presence will often allow for simplification in the metric factors. As a result, we have the following simple combinations for α and β,

$$\begin{aligned}
\nu\rho\,\bar{\alpha}^2 &= -i\,e^\phi \partial_w h_1 - e^{-\phi}\partial_w h_2 \\
\nu\rho\,\bar{\beta}^2 &= +i\,e^\phi \partial_w h_1 - e^{-\phi}\partial_w h_2
\end{aligned} \tag{9.31}$$

and the metric factors are given by,

$$\begin{aligned}
\rho\,f_1 &= -2\nu\,\mathrm{Re}\left(e^{-2\phi}|\partial_w h_2|^2 - e^{2\phi}|\partial_w h_1|^2 - iW\right)^{\frac{1}{2}} \\
\rho\,f_2 &= -2\,\mathrm{Im}\left(e^{-2\phi}|\partial_w h_2|^2 - e^{2\phi}|\partial_w h_1|^2 - iW\right)^{\frac{1}{2}} \\
\rho\,f_4 &= \left|e^{-\phi}\partial_w h_2 - i\,e^\phi \partial_w h_1\right| + \left|e^{-\phi}\partial_w h_2 + i\,e^\phi \partial_w h_1\right|
\end{aligned} \tag{9.32}$$

Note that the following bilinears in f_1 and f_2 are especially simple,

$$\begin{aligned}
\rho^2 f_1 f_2 &= -2\nu W \\
\rho^2 (f_1^2 - f_2^2) &= 4e^{-2\phi}|\partial_w h_2|^2 - 4e^{2\phi}|\partial_w h_1|^2
\end{aligned} \tag{9.33}$$

For completeness, formulas for the metric factors f_1, f_2, and f_4 expressed directly in terms of the harmonic functions h_1 and h_2 are presented in appendix E.

9.8 The 3-form field strengths

To compute the fluxes of the 3-form field strength $F_{(3)} = dB_{(2)}$, it will be useful to also have an explicit expression for the 2-form potential $B_{(2)}$. The form $F_{(3)}$ decomposes into the real NSNS form $H_{(3)}$ and the real RR form $C_{(3)}$,

$$F_{(3)} = H_{(3)} + iC_{(3)} \tag{9.34}$$

Identifying with the form of the Ansatz, in conformal coordinates w on Σ, we have,

$$H_{(3)} = e^{+\phi} g_a e^{45a} = \hat{e}^{45} \wedge e^{+\phi} f_1^2 \left(\rho g_z \, dw + \rho g_{\bar{z}} \, d\bar{w} \right)$$
$$C_{(3)} = e^{-\phi} h_a e^{67a} = \hat{e}^{67} \wedge e^{-\phi} f_2^2 \left(\rho h_z \, dw + \rho h_{\bar{z}} \, d\bar{w} \right) \tag{9.35}$$

The forms $\hat{e}^{45}$ and $\hat{e}^{67}$ are the volume forms on the two unit spheres, as such they are automatically closed forms. Closure of $H_{(3)}$ and $C_{(3)}$ thus requires that,

$$e^{+\phi} f_1^2 \left(\rho g_z \, dw + \rho g_{\bar{z}} \, d\bar{w} \right) = db_1$$
$$e^{-\phi} f_2^2 \left(\rho h_z \, dw + \rho h_{\bar{z}} \, d\bar{w} \right) = db_2 \tag{9.36}$$

for two (locally defined) real functions b_1 and b_2. In order to evaluate line integrals of these currents and compute the associated charges, we calculate b_1 and b_2.

9.8.1 Calculation of $b_{1,2}$

The calculations of b_1 and b_2 proceed in parallel. The starting points are the formulas of (7.1) which express g_z and h_z in terms of α and β. Using the explicit expression for $f_{1,2}$ in terms of α and β, we find,

$$\rho g_z e^{+\phi} f_1^2 = -2(\alpha^2 - \beta^2) \bar{\alpha} \bar{\beta} \, e^{+\phi} \left(\frac{\alpha \bar{\beta}}{\bar{\alpha} \beta} + \frac{\bar{\alpha} \beta}{\alpha \bar{\beta}} + 2 \right) \partial_w \phi$$
$$\rho h_z e^{-\phi} f_2^2 = -2i(\alpha^2 + \beta^2) \bar{\alpha} \bar{\beta} \, e^{-\phi} \left(-\frac{\alpha \bar{\beta}}{\bar{\alpha} \beta} - \frac{\bar{\alpha} \beta}{\alpha \bar{\beta}} + 2 \right) \partial_w \phi \tag{9.37}$$

Using the second equation in (7.3) to eliminate the ratios in the parentheses, and using (7.8) to eliminate the factors $\alpha^2 \pm \beta^2$, and recognizing that all terms in the parenthesis arise from a derivation, we have

$$\rho g_z e^{+\phi} f_1^2 = \partial_w \left(-2\rho^{-1} \bar{\alpha} \bar{\beta} \, \bar{\kappa} \, e^{+2\phi + \bar{\lambda}} \right)$$
$$\rho h_z e^{-\phi} f_2^2 = \partial_w \left(2i \rho^{-1} \bar{\alpha} \bar{\beta} \, \bar{\kappa} \, e^{-2\phi - \bar{\lambda}} \right) \tag{9.38}$$

This result is not yet of the form (9.36) since the arguments of the derivatives are complex. For (9.36) to work, it must be that the imaginary parts of the arguments are actually harmonic. We separate the real and imaginary parts as follows,

$$-\frac{2}{\rho} \bar{\alpha} \bar{\beta} \, \bar{\kappa} \, e^{+2\phi + \bar{\lambda}} = b_{1+} + i b_{1-}$$
$$i \frac{2}{\rho} \bar{\alpha} \bar{\beta} \, \bar{\kappa} \, e^{-2\phi - \bar{\lambda}} = b_{2+} + i b_{2-} \tag{9.39}$$

where $b_{1\pm}$ and $b_{2\pm}$ are real. To calculate b_1 and b_2, we make use of formulas (7.9) to express α and β in terms of ϕ and λ, and formula (9.1) to express the phase in terms of the function ϑ. Using formula (9.8) to further express ρ^2 in terms of $\hat{\rho}^2$, we find the following expressions, from which b_1 and b_2 can be readily evaluated,

$$\frac{1}{\rho} \bar{\alpha} \bar{\beta} \, \bar{\kappa} \, e^{+2\phi + \bar{\lambda}} = -\frac{i}{\hat{\rho}^2} \frac{\cos \frac{1}{2}(\vartheta + \mu)}{\cos \frac{1}{2}(\vartheta - \mu)} e^{-\lambda + i\vartheta/2}$$
$$\frac{1}{\rho} \bar{\alpha} \bar{\beta} \, \bar{\kappa} \, e^{-2\phi - \bar{\lambda}} = -\frac{i}{\hat{\rho}^2} \frac{\sin \frac{1}{2}(\vartheta - \mu)}{\sin \frac{1}{2}(\vartheta + \mu)} e^{+\lambda + i\vartheta/2} \tag{9.40}$$

Putting all together, and using formula (9.12) to express $\hat{\rho}$ and the phase ϑ in terms of the single complex variable ψ, we get

$$
\begin{aligned}
b_{1-} &= \frac{1}{\cos\mu}\left(\psi e^{-\lambda} + \bar{\psi}e^{-\bar{\lambda}}\right) \\
b_{2-} &= \frac{i}{\cos\mu}\left(\psi e^{\lambda} + \bar{\psi}e^{\bar{\lambda}}\right)
\end{aligned}
\tag{9.41}
$$

Finally, using the expression (9.24) for ψ in terms of the harmonic functions h_1 and h_2, we arrive at the final formulas for the imaginary parts,

$$
\begin{aligned}
b_{1-} &= +2h_2 = 2(\mathcal{B} + \bar{\mathcal{B}}) \\
b_{2-} &= -2h_1 = 2i(\mathcal{A} - \bar{\mathcal{A}})
\end{aligned}
\tag{9.42}
$$

Since we have

$$
\begin{aligned}
\partial_w h_1 &= -i\partial_w \tilde{h}_1 & \tilde{h}_1 &= \mathcal{A} + \bar{\mathcal{A}} \\
\partial_w h_2 &= -i\partial_w \tilde{h}_2 & \tilde{h}_2 &= i(\mathcal{B} - \bar{\mathcal{B}})
\end{aligned}
\tag{9.43}
$$

the imaginary parts $ib_{1,2-}$ may be recast in terms of differentials of real functions only,

$$
\begin{aligned}
b_1 &= b_{1+} + 2\tilde{h}_2 \\
b_2 &= b_{2+} - 2\tilde{h}_1
\end{aligned}
\tag{9.44}
$$

The same steps used to compute the imaginary parts may also be used to simplify the real parts and express them solely in terms of the harmonic functions h_1 and h_2. We omit the details of the calculation, and only quote the result,

$$
\begin{aligned}
b_{1+} &= \frac{2ih_1h_2(\partial_w h_1 \partial_{\bar{w}} h_2 - \partial_{\bar{w}} h_1 \partial_w h_2)}{2h_2|\partial_w h_1|^2 - h_1 W} \\
b_{2+} &= \frac{2ih_1h_2(\partial_w h_1 \partial_{\bar{w}} h_2 - \partial_{\bar{w}} h_1 \partial_w h_2)}{2h_1|\partial_w h_2|^2 - h_2 W}
\end{aligned}
\tag{9.45}
$$

The contributions b_{1+} and b_{2+} are well-defined single-valued functions on Σ (since by construction h_1, h_2 are single-valued, as well as their derivatives), but the harmonic duals $\tilde{h}_1$ and $\tilde{h}_2$ are not, generally, single-valued. As a result, the calculation of a flux through a 3-cycle can be greatly simplified. For example, consider a 3-cycle M_3 with the following homological decomposition,

$$
M_3 = C_1 \times (S_1^2)^{n_1} \cup C_2 \times (S_2^2)^{n_2}
\tag{9.46}
$$

where C_1 and C_2 are closed curves in Σ and n_1 and n_2 are integers representing the number of times M_3 contains the spheres S_1^2 and S_2^2. The flux through this cycle is given by

$$
\oint_{M_3} F_{(3)} = 8\pi n_1 \oint_{C_1} d\tilde{h}_2 - 8\pi n_2 i \oint_{C_2} d\tilde{h}_1
\tag{9.47}
$$

The contour integrals over C_1 and C_2 will reduce to residue calculations.

9.9 The 5-form field strength

The conserved flux of the 5-form field strength is given by the cohomology of $dC_{(4)}$. Therefore, we shall compute $C_{(4)}$ for our Ansatz and solutions. The starting point is its expression in terms of $F_{(5)}$ and the 3-form fields,

$$dC_{(4)} = F_{(5)} - \frac{i}{16}\left(B_{(2)} \wedge \bar{F}_{(3)} - \bar{B}_{(2)} \wedge F_{(3)}\right) \tag{9.48}$$

In view of the results obtained in the preceding section, we have

$$
\begin{aligned}
B_{(2)} &= b_1\,\hat{e}^{45} + ib_2\,\hat{e}^{67} \\
F_{(3)} &= e^{\phi}g_a e^a f_1^2 \hat{e}^{45} + ie^{-\phi}h_a e^a f_2^2 \hat{e}^{67} \\
F_{(5)} &= -f_a e^a f_4^4 \hat{e}^{0123} + f_a \varepsilon^{ab} e^b f_1^2 f_2^2 \hat{e}^{4567}
\end{aligned}
\tag{9.49}
$$

From the structure of this Ansatz, it is immediate that $C_{(4)}$ takes on the following form,

$$C_{(4)} = -j_1 \hat{e}^{0123} + j_2 \hat{e}^{4567} \tag{9.50}$$

where the real functions j_1 and j_2 are determined by the differential equations,

$$
\begin{aligned}
dj_1 &= f_a e^a f_4^4 \\
dj_2 &= f_a \varepsilon^{ab} e^b f_1^2 f_2^2 + \frac{1}{8}e^{+\phi}g_a e^a f_1^2 b_2 - \frac{1}{8}e^{-\phi}h_a e^a f_2^2 b_1
\end{aligned}
\tag{9.51}
$$

Closure of these 1-forms on the right hand side of the equations is precisely the contents of the two Bianchi identities for the 5-form, and was established using Mathematica. The form of these functions will not be needed in the sequel, so we shall limit the calculation to j_1.

9.9.1 Calculation of j_1

The starting point is the following expression for ρf_z,

$$2\rho f_z = \left(\frac{\alpha\bar{\beta}}{\bar{\alpha}\beta} - \frac{\bar{\alpha}\beta}{\alpha\bar{\beta}}\right)\partial_w \phi + \partial_w \ln\frac{\bar{\beta}}{\bar{\alpha}} \tag{9.52}$$

It is obtained starting from the two $(a+)$ equations in (6.28), in which $g_z \pm ih_z$ are eliminated using (7.1) and p_z is eliminated from the result using (7.7). The strategy is to first convert this quantity to ϕ, ρ, λ, then to θ, μ and finally to $\psi, \bar{\psi}$ and h_1, h_2. First, we use (7.9) to express the right hand side in terms of ϕ and λ,

$$2\rho f_z = \frac{\partial_w \phi + \partial_w \lambda}{\mathrm{sh}(2\phi + 2\lambda)} - \frac{2\mathrm{sh}(\lambda - \bar{\lambda})\partial_w \phi}{|\mathrm{sh}(2\phi + 2\lambda)|} \tag{9.53}$$

On the other hand, squaring f_4 and expressing the result in terms of ϕ, λ, ρ, we have

$$f_4^2 = \left(|\alpha|^2 + |\beta|^2\right)^2 = \frac{e^{\kappa + \bar{\kappa}}}{\rho^2}\left(\mathrm{ch}(2\phi + \lambda + \bar{\lambda}) + |\mathrm{sh}(2\phi + 2\lambda)|\right) \tag{9.54}$$

Next, we covert these formulas to ϑ, μ, which were defined by (9.1) and $\lambda - \bar{\lambda} = i\mu$, using (9.4). Converting all but the derivatives of ϕ and λ, we find,

$$2\rho f_z f_4^4 = \frac{2\sin(\vartheta - \mu)\sin(\theta + \mu)}{\hat{\rho}^4(\sin\mu)^2} \left[e^{-i\vartheta}(\partial_w\phi + \partial_w\lambda) - 2i(\sin\mu)\partial_w\phi \right] \tag{9.55}$$

Converting now also the derivatives, using (9.5), we get

$$2\rho f_z f_4^4 = \frac{1}{2\hat{\rho}^4(\sin\mu)^2} \Big[e^{-i\vartheta} \left(-(\sin 2\mu)\partial_w\vartheta - ie^{2i\vartheta}\partial_w\mu + i(\cos 2\mu)\partial_w\mu \right) \tag{9.56}$$

$$-2i(\sin\mu) \left(-(\sin 2\mu)\partial_w\vartheta + ie^{-2i\vartheta}\partial_w\mu - i(\cos 2\mu)\partial_w\mu \right) \Big]$$

Converting this expression to $\psi, \bar{\psi}$, using (9.12), we find,

$$2\rho f_z f_4^4 = \frac{2}{(\sin 2\mu)^2} \Big[-i(\sin 2\mu)(\psi\partial_w\psi - \psi^2\partial_w\bar{\psi}/\bar{\psi}) - i\bar{\psi}^2\partial_w\mu \tag{9.57}$$

$$+i\psi^2(\cos 2\mu)\partial_w\mu - 2(\sin\mu)(\sin 2\mu)(\bar{\psi}\partial_w\psi - \psi\partial_w\bar{\psi})$$

$$+2(\sin\mu)\psi^3\partial_w\mu/\bar{\psi} - 2(\sin\mu)(\cos 2\mu)\psi\bar{\psi}\partial_w\mu \Big]$$

Next, we use the field equation for $\partial_w\bar{\psi}$ of (9.13), to eliminate all terms which are not of second order in ψ and/or $\bar{\psi}$. All terms but one may be expressed as total derivatives of a real function,

$$2\rho f_z f_4^4 = \partial_w \left(\frac{i\bar{\psi}^2 - i\psi^2}{\sin 2\mu} - 2\frac{\psi\bar{\psi}}{\cos\mu} \right) + 3i\frac{\psi^2\partial_w\mu}{(\cos\mu)^2} \tag{9.58}$$

The last term may be recast as follows,

$$\frac{\psi^2\partial_w\mu}{(\cos\mu)^2} = \partial_w \left(\psi^2\mathrm{tg}\mu - ih_1^2 e^{-2\bar{\lambda}} - ih_2^2 e^{2\bar{\lambda}} \right) + 2h_1\partial_w h_2 - 2h_2\partial_w h_1 \tag{9.59}$$

where we have used the field equation for $\partial_w\psi$ of (9.13), as well as (9.15) and (9.24) to express κ in terms of the harmonic functions h_1 and h_2. Defining a new holomorphic function, $\mathcal{C}$ by

$$\partial_w\mathcal{C} \equiv \mathcal{A}\partial_w\mathcal{B} - \mathcal{B}\partial_w\mathcal{A} \tag{9.60}$$

allows us to recast the expression in its final form,

$$2\rho f_z f_4^4 = \partial_w \left(6\mathcal{C} + 2h_1 h_2 \mathrm{cotg}\mu \right) \tag{9.61}$$

Here, we have simplified the argument by using the following identity,

$$\frac{i\bar{\psi}^2 - i\psi^2}{\sin 2\mu} - 2\frac{\psi\bar{\psi}}{\cos\mu} + 3i\psi^2\mathrm{tg}\mu + 3h_1^2 e^{-2\bar{\lambda}} + 3h_2^2 e^{2\bar{\lambda}} = 2h_1 h_2 \mathrm{cotg}\mu \tag{9.62}$$

We thus obtain an explicit formula for j_1,

$$j_1 = 3\mathcal{C} + 3\bar{\mathcal{C}} + h_1 h_2 \mathrm{cotg}\mu = 3\mathcal{C} + 3\bar{\mathcal{C}} + ih_1 h_2 \frac{\partial_w h_1 \partial_{\bar{w}} h_2 - \partial_{\bar{w}} h_1 \partial_w h_2}{\partial_w h_1 \partial_{\bar{w}} h_2 + \partial_{\bar{w}} h_1 \partial_w h_2} \tag{9.63}$$

In evaluating closed contour integrals of dj_1, only the $\mathcal{C}$ terms contribute as the other term is single-valued,

$$\oint_C dj_1 = 3 \oint_C d(\mathcal{C} + \bar{\mathcal{C}}) \tag{9.64}$$

9.10 Transformation rules

It will be helpful to derive the effect of simple operations on h_1 and h_2. The first transformation is a constant shift in the dilaton, leaving all other supergravity fields unchanged,

$$h_1 \to e^{-\phi_0} h_1 \qquad h_2 \to e^{\phi_0} h_2 \qquad\qquad \phi \to \phi + \phi_0 \qquad (9.65)$$

The second transformation is a common scaling by a real positive constant Λ^2,

$$h_1 \to \Lambda^2 h_1 \qquad h_2 \to \Lambda^2 h_2 \qquad\qquad (9.66)$$

which transforms the fields as

$$\rho \to \Lambda\rho \qquad f_4 \to \Lambda f_4 \qquad f_{1,2} \to \Lambda f_{1,2}$$
$$g_z \to g_z/\Lambda \qquad h_z \to h_z/\Lambda \qquad f_z \to f_z/\Lambda \qquad (9.67)$$

with all other fields, including the dilaton, left invariant. The third transformation is a strong-weak duality,

$$h_1 \leftrightarrow h_2 \qquad\qquad \phi \to -\phi \qquad\qquad g_z \leftrightarrow h_z \qquad (9.68)$$

Finally, the effect of sign reversal of ν is given by

$$h_{1(2)} \to -h_{1(2)} \qquad h_{2(1)} \to h_{2(1)} \qquad \nu \to -\nu \qquad (9.69)$$

under which the fields transform as

$$f_z \to -f_z \qquad g(h)_z \to -g(h)_z \qquad (9.70)$$

10. The Half-BPS Janus solution

In this section, we shall first recover the $AdS_5 \times S^5$ solution (with constant dilaton and vanishing G-field) from the general $AdS_4 \times S^2 \times S^2 \times \Sigma$ solution derived in the preceding sections. A simple deformation of the harmonic functions h_1 and h_2 of the $AdS_5 \times S^5$ solution will produce a family of regular solutions with varying dilaton and non-zero G-field. This solution is naturally identified with the generalization of the Janus solution that possesses 16 supersymmetries, predicted to exist in [7] on the basis of its dual interface CFT.

10.1 The $AdS_5 \times S^5$ solution

The $AdS_5 \times S^5$ solution has constant dilaton ϕ_0, and is obtained via a linear combination of exponentials with opposite arguments,

$$\mathcal{A} = e^w - e^{-w} \qquad\qquad h_1 = -ie^w + ie^{-w} + ie^{\bar{w}} - ie^{-\bar{w}}$$
$$\mathcal{B} = e^w + e^{-w} \qquad\qquad h_2 = e^w + e^{-w} + e^{\bar{w}} + e^{-\bar{w}} \qquad (10.1)$$

Here, we have used the transformation properties of (9.65) to shift the dilaton to 0 value, and the dilaton equation (9.28) is indeed satisfied with $\phi = 0$; the Σ-metric ρ is constant in these coordinates; and the metric functions are,

$$\rho^2 f_1 f_2 = -8\,i\nu\,\mathrm{sh}(w - \bar{w})$$
$$\rho^2(f_1^2 - f_2^2) = -4\,\mathrm{ch}(w - \bar{w})$$
$$\rho f_4 = 4\,\mathrm{ch}(w + \bar{w}) \tag{10.2}$$

The domain of variation of w may be figured out from the fact that the sphere S^5 arises from $S^2 \times S^2$ varying on an interval with one S^2 vanishing at one end, and the other S^2 vanishing at the other end of the interval. Therefore, the domain must be

$$\Sigma = \left\{ w \in \mathbf{C};\ 0 \leq \mathrm{Im}(w) \leq \frac{\pi}{2} \right\} \tag{10.3}$$

with $\mathrm{Re}(w)$ running over the entire $\mathbf{R}$.

10.2 More general solutions with exponentials

Next, we shall seek regular solutions of the Janus type. This means that the solution will have two asymptotic regions, with the dilaton tending towards distinct constant values $\phi_\pm$ in each asymptotic region. More properly, Janus consists of a family of solutions parametrized by the difference $\phi_+ - \phi_-$ between the dilaton values in these two different regions. This family contains $AdS_5 \times S^5$ for the special value $\phi_+ - \phi_- = 0$.

The behavior of Janus, described above, suggests that Janus should correspond to harmonic functions h_1 and h_2 given by a family of deformations of the harmonic functions of (10.1) of the $AdS_5 \times S^5$ solution. Thus, we are led to seek solutions of the following type,

$$\mathcal{A} = a_+ e^w + a_- e^{-w}$$
$$\mathcal{B} = b_+ e^w + b_- e^{-w} \tag{10.4}$$

The corresponding harmonic functions are given by

$$h_1 = -ia_+ e^w - ia_- e^{-w} + i\bar{a}_+ e^{\bar{w}} + i\bar{a}_- e^{-\bar{w}}$$
$$h_2 = b_+ e^w + b_- e^{-w} + \bar{b}_+ e^{\bar{w}} + \bar{b}_- e^{-\bar{w}} \tag{10.5}$$

These are harmonic functions on the domain Σ without poles. They define a family of solutions to our equations with $AdS_5 \times S^5$ as a special point in this family, corresponding to $a_\pm = \mp e^{-2\phi_0}$ and $b_\pm = 1$. We shall now show that the dilaton tends to a constant as $\mathrm{Re}(w) \to \pm\infty$, under certain conditions on $a_\pm$ and $b_\pm$.

10.2.1 Asymptotics

As $\mathrm{Re}(w) \to \pm\infty$, the leading behavior of the harmonic functions is given by,

$$h_1 \to -ia_\pm e^{\pm w} + i\bar{a}_\pm e^{\pm\bar{w}}$$
$$h_2 \to b_\pm e^{\pm w} + \bar{b}_\pm e^{\pm\bar{w}} \tag{10.6}$$

and the leading behavior of the dilaton is readily evaluated using formula (9.28),

$$e^{4\phi} \to \frac{b_\pm^2 e^{\pm 2(w-\bar{w})} - \bar{b}_\pm^2}{a_\pm^2 e^{\pm 2(w-\bar{w})} - \bar{a}_\pm^2} \tag{10.7}$$

For generic $a_\pm$ and $b_\pm$, this behavior involves a non-trivial dependence on $w - \bar{w}$, even in the limit as $\mathrm{Re}(w) \to \pm\infty$, which is generically singular whenever $e^{w-\bar{w}}$ coincides with the phase of $a_\pm$ or $b_\pm$. If, however, we choose b_+^2/a_+^2 and b_-^2/a_-^2 to be real constants, then the residual dependence cancels, and the limits are regular and constant. Positivity of the exponential on the left hand side requires b_+^2/a_+^2 and b_-^2/a_-^2 be positive.

10.2.2 Restricted family of solutions

We shall solve the asymptoticity conditions, arrived at in the preceding paragraph, as follows,

$$
\begin{aligned}
a_+ &= r_+ b_+ & r_+^2 &= e^{-4\phi_+} \\
a_- &= r_- b_- & r_-^2 &= e^{-4\phi_-}
\end{aligned}
\tag{10.8}
$$

By shifting w by the constant $\frac{1}{2}\ln b_+/b_-$, and defining $b \equiv (b_+ b_-)^{\frac{1}{2}}$, we reduce the harmonic expressions to a simpler form,

$$
\begin{aligned}
h_1 &= -ir_+ b\, e^w - ir_- b\, e^{-w} + ir_+ \bar{b}\, e^{\bar{w}} + ir_- \bar{b}\, e^{-\bar{w}} \\
h_2 &= b\, e^w + b\, e^{-w} + \bar{b}\, e^{\bar{w}} + \bar{b}\, e^{-\bar{w}}
\end{aligned}
\tag{10.9}
$$

We now make use of the transformations (9.65) and (9.66) to further reduce the harmonic functions. Picking $\Lambda^2 = (\sqrt{r_+}|b|)^{-1}$ and $e^{2\phi_0} = r_+$, we are left with the maximally reduced form of the harmonic functions,

$$
\begin{aligned}
h_1 &= -ib\, e^w + irb\, e^{-w} + i\bar{b}\, e^{\bar{w}} - ir\bar{b}\, e^{-\bar{w}} \\
h_2 &= b\, e^w + b\, e^{-w} + \bar{b}\, e^{\bar{w}} + \bar{b}\, e^{-\bar{w}}
\end{aligned}
\tag{10.10}
$$

where we have

$$r = -r_-/r_+ = \pm e^{2\phi_+ - 2\phi_-} \tag{10.11}$$

and we simply take b be its phase, $b = e^{i\varphi}$ for φ real. Dependence on the norm of b may be restored using transformation (9.66). The minus sign has been introduced so that the $AdS_5 \times S^5$ solution with constant dilaton throughout corresponds to $r = +1$, and $b = 1$. The dilaton is given by

$$e^{4\phi - 4\phi_+} = \frac{h_2}{h_1} \times \frac{\partial_w h_2 (h_1 \partial_{\bar{w}} h_2 - h_2 \partial_{\bar{w}} h_1) + c.c.}{\partial_w h_1 (h_2 \partial_{\bar{w}} h_1 - h_1 \partial_{\bar{w}} h_2) + c.c.} \tag{10.12}$$

where we have explicitly included the dilaton shift parameter ϕ_+.

10.3 The Half-BPS Janus solution

Only for the choice $b = 1$ (or, equivalently, $b = -1$) is the above solution free of any singularities, and thus provides a candidate for the Janus solution with maximal supersymmetry. To analyze the regularity properties of the solution, we decompose w into real coordinates,

$$w = x + iy \qquad\qquad x \in \mathbf{R}, \qquad 0 \leq y \leq \frac{\pi}{2} \qquad (10.13)$$

The harmonic functions are given by

$$
\begin{aligned}
h_1 &= 2\sin y \left(e^x + re^{-x}\right) \\
h_2 &= 2\cos y \left(e^x + e^{-x}\right)
\end{aligned}
\qquad (10.14)
$$

and the numerator and denominator of the dilaton solution formula (9.28) take the form,

$$
\begin{aligned}
2h_1 |\partial_w h_2|^2 - h_2 W &= 4N(x,y)\sin y \\
2h_2 |\partial_w h_1|^2 - h_1 W &= 4D(x,y)\cos y
\end{aligned}
\qquad (10.15)
$$

where

$$W = -2(1+r)\sin 2y \qquad (10.16)$$

and the functions N and D are given by,

$$N(x,y) = e^{3x} + re^{-3x} + (1+2r)e^x + (2+r)e^{-x} - (1-r)(e^x - e^{-x})\cos 2y \quad (10.17)$$
$$D(x,y) = e^{3x} + r^2 e^{-3x} + r(1+2r)e^{-x} + (2+r)e^x - (1-r)(e^x - re^{-x})\cos 2y$$

The factors $\cos y$ and $\sin y$ cancel between numerators and denominators in the dilaton formula (9.28), and we are left with

$$e^{4\phi} = e^{2\phi_+ + 2\phi_-} \left(\frac{re^x + re^{-x}}{e^x + re^{-x}}\right) \frac{N(x,y)}{D(x,y)} \qquad (10.18)$$

Note that under $\phi_+ \leftrightarrow \phi_-$, we have $r \leftrightarrow 1/r$. The dilaton solution (10.18) is invariant under this transformation upon simultaneously letting $x \leftrightarrow -x$ and leaving y unchanged. Thus, we may restrict to $|r| \geq 1$ without loss of generality.

10.3.1 Regularity of the dilaton

Clearly, the first factor on the right hand side of (10.18) will be singularity free for all x if and only if $r > 0$, a relation we shall henceforth assume. Combining this with $|r| \geq 1$ from the preceding subsection, regularity thus restricts us to $r \geq 1$. Next, the numerator $N(x,y)$ and denominator $D(x,y)$ will be free of zeros, for any real value of x, y, provided

$$(e^x + e^{-x})^3 + (r-1)\left(e^{-3x} + 2e^x + e^{-x}\right) \geq (r-1)\left|e^x - e^{-x}\right|$$
$$(e^x + e^{-x})^3 + (r-1)\left((r+1)e^{-3x} + (2r+3)e^{-x} + e^x\right) \geq (r-1)\left|e^x - re^{-x}\right| \quad (10.19)$$

Since $e^x + e^{-x} \geq 2$ for all x, and $r \geq 1$, it is sufficient to require that

$$
\begin{aligned}
8 + (r-1)(e^x + e^{-x}) &\geq (r-1) \times \left|e^x - e^{-x}\right| \\
8 + (r-1)(e^x + re^{-x}) &\geq (r-1)\left|e^x - re^{-x}\right|
\end{aligned}
\qquad (10.20)
$$

These inequalities hold for all x and $r \geq 1$. Thus, we conclude that $N(x,y), D(x,y) > 0$
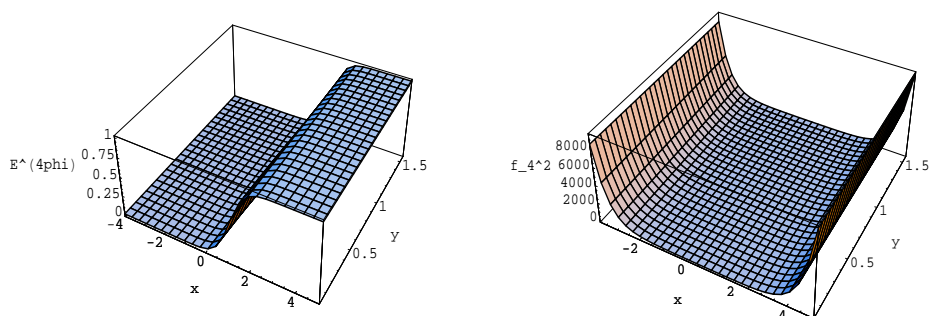
Figure 1: Three-dimensional plot of the dilaton $e^{4\phi}$ (left) and the metric factor f_4^2 (right) for the supersymmetric Janus solution as a function of x and y, for $r = 4$.

10.3.2 Regularity of the metric functions

The Σ-metric factor ρ is given by

$$\rho^8 = \frac{4(1+r)^2 N(x,y)D(x,y)}{(e^x + re^{-x})^3(e^x + e^{-x})^3} \tag{10.21}$$

The S^2 metric factors are given by

$$\rho^2 f_1 f_2 = 4\nu(1+r)\sin 2y \tag{10.22}$$

$$\rho^2(f_1^2 - f_2^2) = -8(1+r)\frac{(e^x + e^{-x})D(x,y)\cos^2 y - (e^x + re^{-x})N(x,y)\sin^2 y}{\sqrt{rN(x,y)D(x,y)(e^x + e^{-x})(e^x + re^{-x})}}$$

These expressions are never singular, as a result of the positivity of $N(x,y)$ and $D(x,y)$. The S^2-metric factors shrink to zero size only on the boundaries of Σ defined by the lines $y = 0$ and $y = \frac{\pi}{2}$. The second equation in (10.22) then shows that f_1 and f_2 cannot simultaneously vanish on $\partial\Sigma$, since either $y = 0$ or $y = \pi/2$ and only a single term survives in the numerator.

The AdS_4 metric factor is given by

$$\rho^2 f_4^2 = 4\sqrt{\frac{N(x,y)D(x,y)}{(e^x + re^{-x})(e^x + e^{-x})}} \tag{10.23}$$

which is non-singular and nowhere vanishing. Taking the limit $r \to 1$, we recover the $AdS_5 \times S^5$ metric factors of (10.2).

10.4 The Half-BPS Janus holographic dual interface theory

The holographic interpretation of the original Janus solution is given by an interface conformal field theory. The dual four-dimensional field theory lives on two four-dimensional half
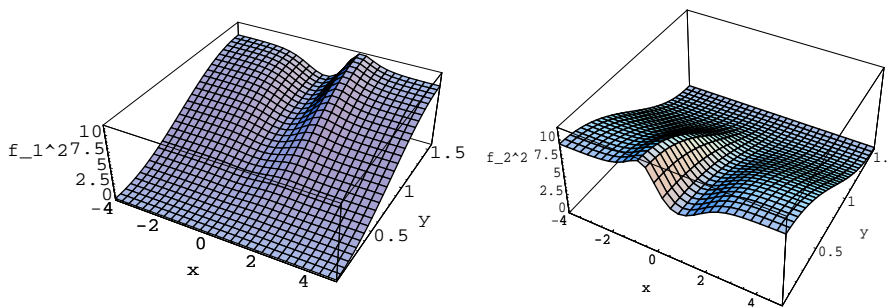
Figure 2: Three-dimensional plot of the metric factor f_1^2 and f_2^2 for the supersymmetric Janus solution as function of x and y, for $r = 4$.

spaces glued together at a three-dimensional interface. Although the Janus solution (10.14) is more complicated than the original [14] and the $\mathcal{N} = 1$ supersymmetric Janus solution [24], it shares many features with these solutions, as we shall show next. The asymptotic behavior of the metric functions can easily be obtained using the parametrization of the strip (10.13). In the limit $x \to \pm\infty$ one gets

$$
\begin{aligned}
\rho^2 &= \sqrt{2}(1+r)^{\frac{1}{2}} + o(e^{-2|x|}) \\
f_1^2 &= 4\sqrt{2}(1+r)^{\frac{1}{2}} \sin^2 y + o(e^{-2|x|}) \\
f_2^2 &= 4\sqrt{2}(1+r)^{\frac{1}{2}} \cos^2 y + o(e^{-2|x|}) \\
f_4^2 &= \frac{2\sqrt{2}}{(1+r)^{\frac{1}{2}}} e^{2|x|} + o(1)
\end{aligned}
\tag{10.24}
$$

The boundary of the bulk geometry can be obtained by extracting the part of the space where the metric becomes infinite. It follows from (10.24) that the metric for AdS_4 blows up when $x \to \pm\infty$. There is however an additional component to the boundary. Employing the Poincaré patch metric for AdS_4.

$$
ds^2_{AdS_4} = \frac{1}{z^2}\Big(- dt^2 + dx_1^2 + dx_2^2 + dz^2 \Big)
\tag{10.25}
$$

it is obvious that there is another boundary component at $z \to 0$. The ten-dimensional asymptotic metric, in the limit $x \to \pm\infty, z \to 0$, is given by[12]

$$
ds^2 \sim \frac{1}{z^2 \mu^2} \Big(z^2 d\mu^2 + \frac{dx_1^2 + dx_2^2 - dt^2 + dz^2}{2(1+r)} \\
+ z^2 \mu^2 (dy^2 + \sin^2 y ds^2_{S_1^2} + \cos^2 y ds^2_{S_2^2}) \Big) + o(\mu^2)
\tag{10.26}
$$

---

[12]Note that the metric on the strip is given by $ds^2_\Sigma = 4\rho^2(dx^2 + dy^2)$ due to our slightly unconventional normalization of the two-dimensional frames $e^z, e^{\bar{z}}$.
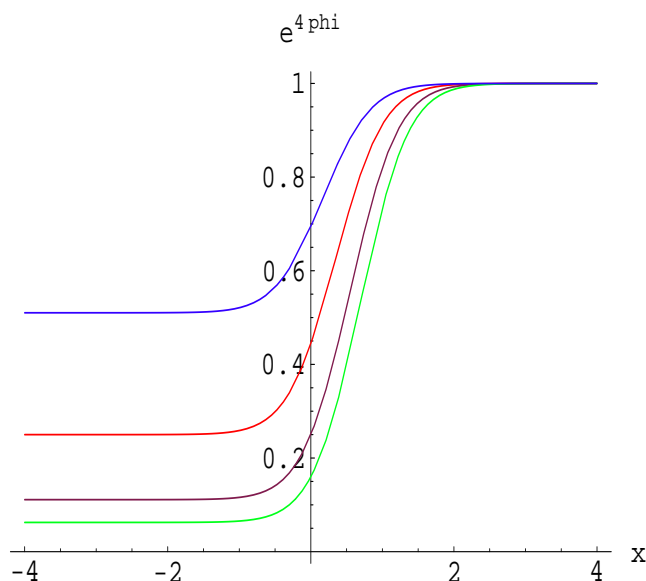
Figure 3: Plot of the dilaton $e^{4\phi}$ at $y = 0$ as a function of x, for $r = 1.5$ (blue), $r = 2$ (red), $r = 3$ (magenta) and $r = 4$ (green).

where a new local coordinate $\mu = e^{\mp x}$ was introduced. The limit $x \to \pm\infty$ corresponds to $\mu \to 0$. It follows from (10.26) that the boundary of the bulk geometry has three components: $x \to \pm\infty$ corresponds to two four-dimensional half spaces, which are glued together at a three-dimensional interface at $z \to 0$. The structure of the boundary is therefore the same as in the original Janus solution and defines an interface field theory. The asymptotic behavior of the dilaton is

$$e^{2\phi} = e^{2\phi_+} + o(e^{-2x}), \text{ as } x \to +\infty$$
$$e^{2\phi} = e^{2\phi_-} + o(e^{-2x}), \text{ as } x \to -\infty, \tag{10.27}$$

Hence the super Yang-Mills theory in the two half spaces has two different values of the coupling constant g_{YM}, as in the original Janus solution. The NSNS and RR 2-form gauge potentials behave as follows near the boundary,

$$\text{Re}(B_{(2)}) \sim e^{-3|x|} \sin y \, \hat{e}^{45}$$
$$\text{Im}(B_{(2)}) \sim e^{-3|x|} \cos y \, \hat{e}^{67} \tag{10.28}$$

Their dependence on the S^5 corresponds to lowest Kaluza-Klein modes on the S^5 of the anti-symmetric rank 2 tensor field and is associated with a scalar field of dimension $\Delta = 3$ [42, 43] in $\mathcal{N} = 4$ super-Yang-Mills.

The behavior (10.28) leads to a insertion of the dual operator which is localized at the interface. This agrees with the interpretation of the solution as a Janus interface CFT with an interface term given by (2.2). The detailed analysis is the same as the one given in [24] and will be repeated in the following for completeness.

In the following we will focus on one of the four-dimensional half spaces and use the local coordinate $\mu = e^{\pm x}$ defined above (not to be confused with the harmonic function μ introduced and used in subsection 9.1). The boundary is reached when $\mu z \to 0$. The complete boundary corresponds to two 4-dimensional half spaces joined by a $\mathbf{R}^3$ interface located at $z = 0$.

The AdS/CFT correspondence relates 10-dimensional Type IIB supergravity fields to gauge invariant operators on the $\mathcal{N} = 4$ super Yang-Mills side. In the following we briefly review some aspects of this map. The Poincaré metric of Euclidean AdS_5 is given by

$$ds^2 = \frac{1}{z^2} \left(dz^2 + \sum_i dx_i^2 \right) \tag{10.29}$$

Near the boundary of AdS_5, where $z \to 0$, a scalar field Φ_m of mass m behaves as,

$$\Phi_m(z, x) \sim \phi_{\text{non-norm}}(x) z^{4-\Delta} + \phi_{\text{norm}}(x) z^{\Delta} \tag{10.30}$$

where $m^2 = \Delta(\Delta - 4)$. The non-normalizable mode corresponds to insertion in the Lagrangian of an operator $\mathcal{O}_\Delta$ with scaling dimension Δ. The boundary source can be determined from (10.30) by

$$\phi_{\text{non-norm}}(x) = \lim_{z \to 0} z^{\Delta-4} \Phi(z, x) \tag{10.31}$$

If $\phi_{non-norm}$ vanishes, a non-zero ϕ_{norm} corresponds to a non-vanishing expectation value

$$\langle \mathcal{O}_\Delta \rangle = \phi_{\text{norm}} \tag{10.32}$$

of the operators $\mathcal{O}_\Delta$ on the Yang-Mills side. The asymptotic behavior near the boundary of the 2-form fields as $\mu \to 0$ is given by

$$\begin{aligned} b_1(\mu) &= \text{const } \mu^3 + o(\mu^5) \\ b_2(\mu) &= \text{const } \mu^3 + o(\mu^5) \end{aligned} \tag{10.33}$$

The state operator correspondence (10.30) seems to suggest that there is no source for the $\Delta = 3$ operator dual to the 3-form fields, since the non-normalizable mode is not turned on. However, this conclusion is premature. For the Janus metric the appropriate rescaling of the field needed to extract the non-normalizable mode is given by

$$\begin{aligned} c_{non-norm} &= \lim_{\epsilon \to 0} \epsilon^{\Delta-4} c(\mu) \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mu^3 \text{ const} \\ &= \lim_{\mu z \to 0} \frac{\mu^2}{z} \text{ const} \end{aligned} \tag{10.34}$$

where $\epsilon = \mu z$ was used. For a point on the boundary which is away from the three-dimensional interface one has $z \neq 0$ and it follows from (10.34) that the source for the dual operator vanishes away from the interface. However for the interface one has $z = 0$ and $c_{non-norm}$ in (10.34) diverges. This behavior indicates the presence of a delta function source for the dual $\Delta = 3$ operator on the interface, since the integral over a small disk around the interface $\int d\mu \, dz z \, c_{non-norm}$ is finite. The localized operator on the interface is the interface counterterm discussed in section 2, which is necessary to restore $\mathcal{N} = 4$ interface supersymmetry.

Acknowledgments

A. Clifford algebra basis adapted to the Ansatz

We choose a basis for the Clifford algebra which is well-adapted to the $AdS_4 \times S_1^2 \times S_2^2 \times \Sigma$ Ansatz, with the frame labeled as in (4.4),

$$
\begin{aligned}
\Gamma^m &= \gamma^m \otimes I_2 \otimes I_2 \otimes I_2 & m &= 0, 1, 2, 3 \\
\Gamma^{i_1} &= \gamma_{(1)} \otimes \gamma^{i_1} \otimes I_2 \otimes I_2 & i_1 &= 4, 5 \\
\Gamma^{i_2} &= \gamma_{(1)} \otimes \sigma^3 \otimes \gamma^{i_2} \otimes I_2 & i_2 &= 6, 7 \\
\Gamma^a &= \gamma_{(1)} \otimes \sigma^3 \otimes \sigma^3 \otimes \gamma^a & a &= 8, 9
\end{aligned}
\tag{A.1}
$$

where a convenient basis for the lower dimensional Clifford algebras is as follows,

$$
\begin{aligned}
i\gamma^0 &= \sigma^2 \otimes I_2 & \gamma^4 = \gamma^6 = \gamma^8 &= \sigma^1 \\
\gamma^1 &= \sigma^1 \otimes I_2 & \gamma^5 = \gamma^7 = \gamma^9 &= \sigma^2 \\
\gamma^2 &= \sigma^3 \otimes \sigma^2 & & \\
\gamma^3 &= \sigma^3 \otimes \sigma^1 & &
\end{aligned}
\tag{A.2}
$$

We shall also need the chirality matrices on the various components of $AdS_4 \times S_1^2 \times S_2^2 \times \Sigma$, and they are chosen as follows,

$$
\begin{aligned}
\gamma_{(1)} &= -i\Gamma^{0123} = \sigma^3 \otimes \sigma^3 \otimes I_2 \otimes I_2 \otimes I_2 \\
\gamma_{(2)} &= -i\Gamma^{45} = I_2 \otimes I_2 \otimes \sigma^3 \otimes I_2 \otimes I_2 \\
\gamma_{(3)} &= -i\Gamma^{67} = I_2 \otimes I_2 \otimes I_2 \otimes \sigma^3 \otimes I_2 \\
\gamma_{(4)} &= -i\Gamma^{89} = I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma^3
\end{aligned}
\tag{A.3}
$$

The 10-dimensional chirality matrix in this basis is given by

$$
\Gamma^{11} = \Gamma^{0123456789} = \gamma_{(1)}\gamma_{(2)}\gamma_{(3)}\gamma_{(4)}
\tag{A.4}
$$

The complex conjugation matrices in each component are defined by

$$
\begin{aligned}
(\gamma^m)^* &= +B_{(1)}\gamma^m B_{(1)}^{-1} & (B_{(1)})^* B_{(1)} &= +I_2 & B_{(1)} &= i\gamma_{(1)}\gamma^2 \\
(\gamma^{i_1})^* &= -B_{(2)}\gamma^{i_1} B_{(2)}^{-1} & (B_{(2)})^* B_{(2)} &= -I_2 & B_{(2)} &= \gamma^5 \\
(\gamma^{i_2})^* &= -B_{(3)}\gamma^{i_2} B_{(3)}^{-1} & (B_{(3)})^* B_{(3)} &= -I_2 & B_{(3)} &= \gamma^7 \\
(\gamma^a)^* &= -B_{(4)}\gamma^a B_{(4)}^{-1} & (B_{(4)})^* B_{(4)} &= -I_2 & B_{(4)} &= \gamma^9
\end{aligned} \tag{A.5}
$$

where in the last column we have also listed the form of these matrices in our particular basis. The 10-dimensional complex conjugation matrix $\mathcal{B}$ is defined by $(\Gamma^M)^* = \mathcal{B}\Gamma^M\mathcal{B}^{-1}$ and $\mathcal{B}\mathcal{B}^* = I$, and in this basis is given by

$$
\mathcal{B} = -\Gamma^{2579} = i\gamma_{(1)}\gamma^2 \otimes \gamma^5 \otimes \gamma^6 \otimes \gamma^9 = iB_{(1)} \otimes B_{(2)} \otimes \left(\gamma_{(3)}B_{(3)}\right) \otimes B_{(4)} \tag{A.6}
$$

B. The geometry of Killing spinors

In this appendix, we review the relation between the Killing spinor equation and the parallel transport equation in the presence of a flat connection with torsion on S^2 and AdS_4.

B.1 The sphere S^2

On S^2, the Killing spinor equation is given by,

$$
\left(\nabla_i + \eta\frac{1}{2}\sigma_i\sigma_3\right)\varepsilon = 0 \qquad\qquad i = 1, 2 \tag{B.1}
$$

where $\sigma^1, \sigma^2, \sigma^3$ are the standard Pauli matrices, and ∇_i is the spin connection on S^2. (An equivalent equation is obtained by letting $\sigma_i\sigma_3 \to i\sigma_i$ and $\varepsilon' = e^{-i\pi\sigma_3/4}\varepsilon$.) Integrability of this system of equations on the round sphere requires $\eta = \pm 1$.

The relevant flat connection with torsion is given by the Maurer-Cartan forms $\omega^{(t)}$ on SO(3), in the spinor representation of SO(3),

$$
\omega^{(t)} = U^\dagger dU = \frac{1}{4}\omega_{IJ}^{(t)}\,\sigma^{[I}\sigma^{J]} \qquad\qquad U \in \text{SU}(2) \tag{B.2}
$$

$$
d\omega^{(t)} + \omega^{(t)} \wedge \omega^{(t)} = 0 \tag{B.3}
$$

where $\{\sigma_I, \sigma_J\} = 2\delta_{IJ}$, with $I, J = 1, 2, 3$. The Maurer-Cartan equation (B.3) expresses the flatness of the connection $\omega^{(t)}$, which in turn reflects the fact that SO(3), like any Lie group, is paralellizable. Next, we view S^2 as the coset space $S^2 = \text{SO}(3)/\text{SO}(2)$, and decompose the directions of the cotangent space of SO(3) accordingly,

$$
\omega^{(t)} = \frac{i}{2}\omega\sigma_3 + \frac{1}{2}e_i\sigma_i\sigma_3 \qquad
\begin{cases}
\omega \equiv \omega_{12}^{(t)} \\
e_i \equiv \omega_{i3}^{(t)} \qquad i = 1, 2
\end{cases} \tag{B.4}
$$

where e_i is the canonical frame, and ω the canonical SO(2) connection on S^2. The Maurer-Cartan equations imply the absence of torsion, and the constancy of curvature. The parallel transport equation for a 2-component spinor ε_+ is simply

$$
\left(d + \omega^{(t)}\right)\varepsilon_+ = 0 \tag{B.5}
$$

On the one hand, using the identification $\omega^{(t)} = U^\dagger dU$, it may be solved trivially by

$$\varepsilon_+ = U^\dagger \varepsilon_0 \qquad\qquad (B.6)$$

where ε_0 is a constant spinor. On the other hand, using the canonical decomposition (B.4), and the expression for the covariant derivative in terms of forms, $e_i \nabla_i = d + i\omega\,\sigma_3/2$, it is clear that the equation coincides with the Killing spinor equation with $\eta = +1$. The solution ε_- to the Killing equation for $\eta = -1$ may be obtained from the solution ε_+, by $\varepsilon_- = \sigma_3\,\varepsilon_+$.

B.2 Minkowski AdS_4

The above construction may be generalized to all spheres and their hyperbolic counterparts. Here, we present the case of Minkowski signature $AdS_4 = \mathrm{SO}(2,3)/\mathrm{SO}(1,3)$. The Clifford algebra of $\mathrm{SO}(2,3)$ is built from the Clifford generators γ^μ, of the Lorentz group $\mathrm{SO}(1,3)$,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \qquad\qquad \eta = \mathrm{diag}[- + + +] \qquad\qquad (B.7)$$

for $\mu, \nu = 0, 1, 2, 3$, supplemented with the chirality matrix, $\gamma^\sharp \equiv \gamma^{0123}$,

$$\{\gamma^{\bar\mu}, \gamma^{\bar\nu}\} = 2\bar\eta^{\bar\mu\bar\nu} \qquad\qquad \bar\eta = \mathrm{diag}[- - + + +] \qquad\qquad (B.8)$$

for $\bar\mu, \bar\nu = \sharp, 0, 1, 2, 3$. The corresponding Maurer-Cartan form on $\mathrm{SO}(2,3)$ is given by

$$\omega^{(t)} = V^{-1}dV = \frac{1}{4}\omega^{(t)}_{\bar\mu\bar\nu}\gamma^{\bar\mu\bar\nu} \qquad\qquad V \in Sp(2,2) \sim \mathrm{SO}(2,3) \qquad\qquad (B.9)$$

It obviously satisfies the Maurer-Cartan equations, $d\omega^{(t)} + \omega^{(t)} \wedge \omega^{(t)} = 0$. We decompose $\omega^{(t)}$ onto the $\mathrm{SO}(1,3)$ and AdS_4 directions of cotangent space,

$$\omega^{(t)} = \frac{1}{4}\omega_{\mu\nu}\gamma^{\mu\nu} + \frac{i}{2}e_\mu\gamma^\mu\gamma_{(4)} \qquad\qquad \begin{cases} \omega_{\mu\nu} \equiv \omega^{(t)}_{\mu\nu} & \mu, \nu = 0, 1, 2, 3 \\ e_\mu \equiv \omega^{(t)}_{\mu\sharp} & \mu = 0, 1, 2, 3 \end{cases} \qquad (B.10)$$

The Maurer-Cartan equations $d\omega^{(t)} + \omega^{(t)} \wedge \omega^{(t)} = 0$ for $\omega^{(t)}$ imply that the absence of torsion and that the constancy of curvature. The Killing spinor equation coincides with the equation for parallel transport,

$$\left(d + V^{-1}dV\right)\varepsilon = \left(d + \frac{1}{4}\omega_{\mu\nu}\gamma^{\mu\nu} + \frac{i}{2}\eta e_\mu\gamma^\mu\gamma_{(4)}\right)\varepsilon = 0 \qquad\qquad (B.11)$$

For $\eta = +1$, the general solution is given by $\varepsilon_+ = V^{-1}\varepsilon_0$ and ε_0 is constant, while for $\eta = -1$, the solution is $\varepsilon_- = \gamma^\sharp\varepsilon_+$. Note that V is a symplectic matrix, so that $V^t J V = J$, and this allows us to define an invariant inner product on the spinors.

C. The derivation of the BPS equations

We begin by collecting some identities that will be useful during the reduction of the BPS equations over the Ansatz of subsection 4.1. The Clifford algebra matrices needed are,

$$
\begin{aligned}
\Gamma^a &= \gamma_{(1)}\gamma_{(2)}\gamma_{(3)}\gamma^a \\
\Gamma^{45a} &= i\gamma_{(1)}\gamma_{(3)}\gamma^a \\
\Gamma^{67a} &= i\gamma_{(1)}\gamma_{(2)}\gamma^a \\
\Gamma^{0123a} &= i\gamma_{(2)}\gamma_{(3)}\gamma^a \\
\varepsilon^{ab}\Gamma^{4567b} &= -\gamma_{(1)}\varepsilon^{ab}\gamma^b
\end{aligned}
\tag{C.1}
$$

We shall also need the following decompositions of ε and $\mathcal{B}^{-1}\varepsilon^*$,

$$
\begin{aligned}
\varepsilon &= \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3} \otimes \zeta_{\eta_1,\eta_2,\eta_3} \\
\mathcal{B}^{-1}\varepsilon^* &= \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3} \otimes *\zeta_{\eta_1,\eta_2,\eta_3}
\end{aligned}
\tag{C.2}
$$

where we use the abbreviation,

$$
\begin{aligned}
*\zeta_{\eta_1,\eta_2,\eta_3} &= -i\sigma^2 \eta_1\eta_2\eta_3 \zeta^*_{\eta_1,\eta_2,-\eta_3} \\
*\zeta &= \tau^{332} \otimes \sigma^2 \zeta^*
\end{aligned}
\tag{C.3}
$$

in τ-matrix notation.

C.1 The dilatino equation

The dilatino equation is,

$$
0 = iP_A\Gamma^A\mathcal{B}^{-1}\varepsilon^* - \frac{i}{24}\Gamma\cdot G\varepsilon
\tag{C.4}
$$

Reduced to the Ansatz of subsection 4.1, we have the following simplifications,

$$
\begin{aligned}
P_A\Gamma^A &= p_a\Gamma^a \\
\Gamma\cdot G &= 3!(g_a\Gamma^{45a} + i\,h_a\Gamma^{67a})
\end{aligned}
\tag{C.5}
$$

The dilatino equation now becomes,

$$
\begin{aligned}
0 = {}& ip_a\Gamma^a \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3}(-i\sigma^2)\eta_1\eta_2\eta_3\zeta^*_{\eta_1,\eta_2,-\eta_3} \\
& -\frac{i}{4}(g_a\Gamma^{45a} + i\,h_a\Gamma^{67a})\sum_{\eta_1,\eta_2,\eta_3}\chi^{\eta_1,\eta_2,\eta_3}\otimes\zeta_{\eta_1,\eta_2,\eta_3}
\end{aligned}
\tag{C.6}
$$

Using the above form of the Γ-matrices, the action of the chirality matrices on χ, and flipping the signs in the summation over η so as to have a common factor of χ, we obtain,

$$
0 = \sum_{\eta_1,\eta_2,\eta_3}\chi^{\eta_1,\eta_2,\eta_3}\otimes\left(-p_a\sigma^a\sigma^2\eta_1\eta_2\eta_3\zeta^*_{-\eta_1,-\eta_2,\eta_3}+\frac{1}{4}g_a\sigma^a\zeta_{-\eta_1,\eta_2,-\eta_3}+\frac{i}{4}h_a\sigma^a\zeta_{-\eta_1,-\eta_2,\eta_3}\right)
\tag{C.7}
$$

Since the $\chi^{\eta_1, \eta_2, \eta_3}$ are linearly independent we require the vanishing of

$$0 = -p_a \sigma^a \sigma^2 \eta_1 \eta_2 \eta_3 \zeta^*_{-\eta_1, -\eta_2, \eta_3} + \frac{1}{4} g_a \sigma^a \zeta_{-\eta_1, \eta_2, -\eta_3} + \frac{i}{4} h_a \sigma^a \zeta_{-\eta_1, -\eta_2, \eta_3} \qquad \text{(C.8)}$$

This can be recast economically using the τ-matrix notation,

$$0 = p_a \tau^{(223)} \sigma^a \sigma^2 \zeta^* + \frac{1}{4} g_a \tau^{(101)} \sigma^a \zeta + \frac{i}{4} h_a \tau^{(110)} \sigma^a \zeta \qquad \text{(C.9)}$$

Upon multiplication on the left by $\tau^{(223)}$, we recover (5.9).

C.2 The gravitino equation

The gravitino equation is

$$
\begin{aligned}
0 &= d\varepsilon + \omega \varepsilon + \phi \varepsilon + g \mathcal{B}^{-1} \varepsilon^* \\
\omega &= \frac{1}{4} \omega_{AB} \Gamma^{AB} \\
\phi &= -\frac{i}{2} Q + \frac{i}{480} (\Gamma \cdot F_{(5)}) e_A \Gamma^A \\
g &= -\frac{1}{96} e_A \left(\Gamma^A (\Gamma \cdot G) + 2 (\Gamma \cdot G) \Gamma^A \right)
\end{aligned}
\qquad \text{(C.10)}
$$

C.2.1 The calculation of ω

The spin connection components are $\omega^a{}_b$, whose explicit form we shall not need, and

$$
\begin{aligned}
\omega^m{}_n &= \hat{\omega}^m{}_n & \omega^m{}_a &= e^m \frac{\partial_a f_4}{f_4} \\
\omega^{i_1}{}_{i_2} &= \hat{\omega}^{i_1}{}_{i_2} & \omega^{i_1}{}_a &= e^{i_1} \frac{\partial_a f_1}{f_1} \\
\omega^{i_2}{}_{j_2} &= \hat{\omega}^{i_2}{}_{j_2} & \omega^{i_2}{}_a &= e^{i_2} \frac{\partial_a f_2}{f_2}
\end{aligned}
\qquad \text{(C.11)}
$$

The hats refers to the canonical connections on AdS_4, S_1^2, S_2^2 respectively. Projecting the spin-connection along the various directions we have

$$
\begin{aligned}
(m) &\qquad \nabla'_m \varepsilon + \frac{1}{2} \frac{D_a f_4}{f_4} \Gamma_m \Gamma^a \varepsilon \\
(i_1) &\qquad \nabla'_{i_1} \varepsilon + \frac{1}{2} \frac{D_a f_1}{f_1} \Gamma_{i_1} \Gamma^a \varepsilon \\
(i_2) &\qquad \nabla'_{i_2} \varepsilon + \frac{1}{2} \frac{D_a f_2}{f_2} \Gamma_{i_2} \Gamma^a \varepsilon \\
(a) &\qquad \nabla_a \varepsilon
\end{aligned}
\qquad \text{(C.12)}
$$

where the prime on the covariant derivative indicates that only the connection along AdS_4, S_1^2 or S_2^2 respectively is included. Using the Killing spinor equations (5.1) we can eliminate

the primed covariant derivatives, which yields

$$(m) \qquad \frac{1}{2f_4}\Gamma_m \sum_{\eta_1,\eta_2,\eta_3} \eta_1 \chi^{\eta_1,\eta_2,\eta_3} \otimes \zeta_{\eta_1,\eta_2,\eta_3} + \frac{1}{2}\frac{D_a f_4}{f_4}\Gamma_m\Gamma^a\varepsilon$$

$$(i_1) \qquad \frac{i}{2f_1}\Gamma_{i_1}\gamma_{(1)} \sum_{\eta_1,\eta_2,\eta_3} \eta_2 \chi^{\eta_1,\eta_2,\eta_3} \otimes \zeta_{\eta_1,\eta_2,\eta_3} + \frac{1}{2}\frac{D_a f_1}{f_1}\Gamma_{i_1}\Gamma^a\epsilon$$

$$(i_2) \qquad \frac{i}{2f_2}\Gamma_{i_2}\gamma_{(1)}\gamma_{(2)} \sum_{\eta_1,\eta_2,\eta_3} \eta_3 \chi^{\eta_1,\eta_2,\eta_3} \otimes \zeta_{\eta_1,\eta_2,\eta_3} + \frac{1}{2}\frac{D_a f_2}{f_2}\Gamma_{i_2}\Gamma^a\varepsilon \qquad (\text{C.13})$$

Using the equation $\Gamma^a = \gamma_{(1)}\gamma_{(2)}\gamma_{(3)}\sigma^a$, we have

$$(m) \qquad \Gamma_m \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3} \otimes \left(\frac{1}{2f_4}\eta_1\zeta_{\eta_1,\eta_2,\eta_3} + \frac{1}{2}\frac{D_a f_4}{f_4}\sigma^a\zeta_{-\eta_1,-\eta_2,-\eta_3} \right)$$

$$(i_1) \qquad \Gamma_{i_1} \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3} \otimes \left(\frac{i}{2f_1}\eta_2\zeta_{-\eta_1,\eta_2,\eta_3} + \frac{1}{2}\frac{D_a f_1}{f_1}\sigma^a\zeta_{-\eta_1,-\eta_2,-\eta_3} \right)$$

$$(i_2) \qquad \Gamma_{i_2} \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3} \otimes \left(\frac{i}{2f_2}\eta_3\zeta_{-\eta_1,-\eta_2,\eta_3} + \frac{1}{2}\frac{D_a f_2}{f_2}\sigma^a\zeta_{-\eta_1,-\eta_2,-\eta_3} \right) \quad (\text{C.14})$$

where we have pulled a factor of Γ_M out front. It will turn out that all terms in the gravitino equation contain $\Gamma_M\chi^{\eta_1,\eta_2,\eta_3}$, and we will require the coefficients to vanish independently, just as we did for the dilatino equation. The coefficient of $\Gamma_M\chi^{\eta_1,\eta_2,\eta_3}$ can be expressed in the τ-matrix notation as

$$(m) \qquad \frac{1}{2f_4}\tau^{(300)}\zeta + \frac{1}{2}\frac{D_a f_4}{f_4}\tau^{(111)}\sigma^a\zeta$$

$$(i_1) \qquad \frac{i}{2f_1}\tau^{(130)}\zeta + \frac{1}{2}\frac{D_a f_1}{f_1}\tau^{(111)}\sigma^a\zeta$$

$$(i_2) \qquad \frac{i}{2f_2}\tau^{(113)}\zeta + \frac{1}{2}\frac{D_a f_2}{f_2}\tau^{(111)}\sigma^a\zeta \qquad (\text{C.15})$$

C.2.2 The calculation of ϕ

The Q part is trivial. The $F_{(5)}$ part is

$$\frac{i}{480}(\Gamma \cdot F_{(5)})\Gamma^A e_A\varepsilon = -\frac{i}{2}f_a\Gamma^{0123a}\Gamma^A e_A\varepsilon \qquad (\text{C.16})$$

Projecting along the various directions, we have

$$(m) \qquad \Gamma_m \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3} \otimes \frac{1}{2}f_a\sigma^a\zeta_{\eta_1,-\eta_2,-\eta_3}$$

$$(i_1) \qquad -\Gamma_{i_1} \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3} \otimes \frac{1}{2}f_a\sigma^a\zeta_{\eta_1,-\eta_2,-\eta_3}$$

$$(i_2) \qquad -\Gamma_{i_2} \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3} \otimes \frac{1}{2}f_a\sigma^a\zeta_{\eta_1,-\eta_2,-\eta_3}$$

$$(a) \qquad \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3} \otimes \left(-\frac{iQ}{2}\zeta_{\eta_1,\eta_2,\eta_3} + \frac{1}{2}f_b\sigma^b\sigma_a\zeta_{-\eta_1,\eta_2,\eta_3} \right) \qquad (\text{C.17})$$

Using the τ-matrix notation, we can write the coefficient of $\Gamma_M \chi^{\eta_1,\eta_2,\eta_3}$ in the form

$$(m) \qquad \frac{1}{2} f_a \tau^{(011)} \otimes \sigma^a \zeta \tag{C.18}$$

$$(i_1) \qquad -\frac{1}{2} f_a \tau^{(011)} \otimes \sigma^a \zeta$$

$$(i_2) \qquad -\frac{1}{2} f_a \tau^{(011)} \otimes \sigma^a \zeta$$

$$(a) \qquad -\frac{i q_a}{2} \zeta + \frac{1}{2} f_b \tau^{(100)} \otimes \sigma^b \sigma_a \zeta$$

where q_a is defined by $Q = q_a e^a$.

C.2.3 The calculation of g

The relevant expression is as follows,

$$g\mathcal{B}^{-1}\epsilon^* = -\frac{3!}{96} e_A \bigg(g_a(\Gamma^A \Gamma^{45a} + 2\Gamma^{45a}\Gamma^A) + i h_a(\Gamma^A \Gamma^{67a} + 2\Gamma^{67a}\Gamma^A) \bigg) \mathcal{B}^{-1}\epsilon^* \tag{C.19}$$

A few useful equations are as follows,

$$\Gamma^a \Gamma^{45b} + 2\Gamma^{45b}\Gamma^a = \Gamma^{45}(3\delta^{ab} - \Gamma^{ab}) = i\gamma_{(2)}(3\delta^{ab} - \sigma^{ab})$$
$$\Gamma^a \Gamma^{67b} + 2\Gamma^{67b}\Gamma^a = \Gamma^{67}(3\delta^{ab} - \Gamma^{ab}) = i\gamma_{(3)}(3\delta^{ab} - \sigma^{ab}) \tag{C.20}$$

Projecting along the various directions we obtain

$$(m) \qquad -\Gamma_m \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3} \otimes \frac{1}{16}(-ig_a\sigma^a * \zeta_{-\eta_1,\eta_2,-\eta_3} + h_a\sigma^a * \zeta_{-\eta_1,-\eta_2,\eta_3}) \tag{C.21}$$

$$(i_1) \qquad -\Gamma_{i_1} \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3} \otimes \frac{1}{16}(3ig_a\sigma^a * \zeta_{-\eta_1,\eta_2,-\eta_3} + h_a\sigma^a * \zeta_{-\eta_1,-\eta_2,\eta_3})$$

$$(i_2) \qquad -\Gamma_{i_2} \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3} \otimes \frac{1}{16}(-ig_a\sigma^a * \zeta_{-\eta_1,\eta_2,-\eta_3} - 3h_a\sigma^a * \zeta_{-\eta_1,-\eta_2,\eta_3})$$

$$(a) \qquad -\sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3} \otimes \frac{1}{16}\bigg(3ig_a * \zeta_{\eta_1,-\eta_2,\eta_3} - ig_b\sigma^{ab} * \zeta_{\eta_1,-\eta_2,\eta_3}$$

$$-3h_a * \zeta_{\eta_1,\eta_2,-\eta_3} + h_a\sigma^{ab} * \zeta_{\eta_1,\eta_2,-\eta_3} \bigg)$$

Using the τ-matrix notation, we can write the coefficient of $\Gamma_M \chi^{\eta_1,\eta_2,\eta_3}$ in the form

$$(m) \quad -\frac{1}{16}(-ig_a\tau^{(101)}\sigma^a * \zeta + h_a\tau^{(110)}\sigma^a * \zeta) \tag{C.22}$$

$$(i_1) \quad -\frac{1}{16}(3ig_a\tau^{(101)}\sigma^a * \zeta + h_a\tau^{(110)}\sigma^a * \zeta)$$

$$(i_2) \quad -\frac{1}{16}(-ig_a\tau^{(101)}\sigma^a * \zeta - 3h_a\tau^{(110)}\sigma^a * \zeta)$$

$$(a) \quad -\frac{1}{16}\bigg(3ig_a\tau^{(010)} * \zeta - ig_b\tau^{(010)}\sigma^{ab} * \zeta - 3h_a\tau^{(001)} * \zeta + h_a\tau^{(001)}\sigma^{ab} * \zeta \bigg)$$

C.2.4 Assembling the complete gravitino BPS equation

Now we combine the three equations $(C.15)$, $(C.18)$, and $(C.22)$ to obtain the reduced gravitino equations. We again argue that the $\Gamma_M \chi^{\eta_1\eta_2\eta_3}$ are linearly independent which leads to the equations

$$
\begin{aligned}
(m) \qquad 0 &= \frac{1}{2f_4}\tau^{(300)}\zeta + \frac{1}{2}\frac{D_a f_4}{f_4}\tau^{(111)}\sigma^a\zeta + \frac{1}{2}f_a\tau^{(011)}\sigma^a\zeta \\
&\quad + \frac{1}{16}ig_a\tau^{(101)}\sigma^a *\zeta - \frac{1}{16}h_a\tau^{(110)}\sigma^a *\zeta \\[4pt]
(i_1) \qquad 0 &= \frac{i}{2f_1}\tau^{(130)}\zeta + \frac{1}{2}\frac{D_a f_1}{f_1}\tau^{(111)}\sigma^a\zeta - \frac{1}{2}f_a\tau^{(011)}\sigma^a\zeta \\
&\quad - \frac{3}{16}ig_a\tau^{(101)}\sigma^a *\zeta - \frac{1}{16}h_a\tau^{(110)}\sigma^a *\zeta \\[4pt]
(i_2) \qquad 0 &= \frac{i}{2f_2}\tau^{(113)}\zeta + \frac{1}{2}\frac{D_a f_2}{f_2}\tau^{(111)}\sigma^a\zeta - \frac{1}{2}f_a\tau^{(011)}\sigma^a\zeta \\
&\quad + \frac{1}{16}ig_a\tau^{(101)}\sigma^a *\zeta + \frac{3}{16}h_a\tau^{(110)}\sigma^a *\zeta \\[4pt]
(a) \qquad 0 &= \left(D_a + \frac{i}{2}\hat{\omega}_a\sigma^3\right)\zeta - \frac{i}{2}q_a\zeta + \frac{1}{2}f_b\tau^{(100)}\sigma^b\sigma_a\zeta \\
&\quad - \frac{3}{16}ig_a\tau^{(010)}*\zeta + \frac{1}{16}ig_b\tau^{(010)}\sigma^{ab}*\zeta \\
&\quad + \frac{3}{16}h_a\tau^{(001)}*\zeta - \frac{1}{16}h_b\tau^{(001)}\sigma^{ab}*\zeta
\end{aligned} \tag{C.23}
$$

where $\hat{\omega}_a = (\hat{\omega}_{89})_a$ is the spin connection along Σ. In the first three equations, we have dropped an overall factor of Γ_M. In the last equation, we have used the connection formula $(C.11)$ for the covariant derivative and the fact $\Gamma^{89} = i\,\sigma^3$. Eliminating the star using the definition $(C.3)$, $*\zeta = \tau^{(332)}\otimes\sigma^2\zeta^*$. The system of gravitino BPS equations is then

$$
\begin{aligned}
(m) \qquad 0 &= \frac{1}{2f_4}\tau^{(300)}\zeta + \frac{1}{2}\frac{\partial_a f_4}{f_4}\tau^{(111)}\sigma^a\zeta + \frac{1}{2}f_a\tau^{(011)}\sigma^a\zeta \\
&\quad + \frac{1}{16}ig_a\tau^{(233)}\sigma^a\sigma^2\zeta^* + \frac{1}{16}h_a\tau^{(222)}\sigma^a\sigma^2\zeta^* \\[4pt]
(i_1) \qquad 0 &= \frac{i}{2f_1}\tau^{(130)}\zeta + \frac{1}{2}\frac{\partial_a f_1}{f_1}\tau^{(111)}\sigma^a\zeta - \frac{1}{2}f_a\tau^{(011)}\sigma^a\zeta \\
&\quad - \frac{3}{16}ig_a\tau^{(233)}\sigma^a\sigma^2\zeta^* + \frac{1}{16}h_a\tau^{(222)}\sigma^a\sigma^2\zeta^* \\[4pt]
(i_2) \qquad 0 &= \frac{i}{2f_2}\tau^{(113)}\zeta + \frac{1}{2}\frac{\partial_a f_2}{f_2}\tau^{(111)}\sigma^a\zeta - \frac{1}{2}f_a\tau^{(011)}\sigma^a\zeta \\
&\quad + \frac{1}{16}ig_a\tau^{(233)}\sigma^a\sigma^2\zeta^* - \frac{3}{16}h_a\tau^{(222)}\sigma^a\sigma^2\zeta^* \\[4pt]
(a) \qquad 0 &= \left(\partial_a I_8 + \frac{i}{2}\hat{\omega}_a\sigma^3\right)\zeta - \frac{i}{2}q_a\zeta + \frac{1}{2}f_b\tau^{(100)}\sigma^b\sigma_a\zeta \\
&\quad - \frac{3}{16}g_a\tau^{(322)}\sigma^2\zeta^* + \frac{1}{16}g_b\tau^{(322)}\sigma^{ab}\sigma^2\zeta^* \\
&\quad + \frac{3}{16}ih_a\tau^{(333)}\sigma^2\zeta^* - \frac{1}{16}ih_b\tau^{(333)}\sigma^{ab}\sigma^2\zeta^*
\end{aligned} \tag{C.24}
$$

Upon multiplying the (m), (i_1) and (i_2) equations by $\tau^{(111)}$, we recover (5.10).

D. Solving the bilinear constraints

In view of the chirality constraint, certain bilinears, with insertions of an arbitrary Hermitian τ-matrix M, and arbitrary σ-matrix σ^p, for $p = 0, 1, 2, 3$, vanish automatically,

$$\zeta^\dagger M \sigma^p \zeta = 0, \qquad \text{if} \qquad \left\{ M\sigma^p, \tau^{(111)}\sigma^3 \right\} = 0 \tag{D.1}$$

D.1 Construction of vanishing bilinears

We form bilinears with the help of τ-matrices T such that

$$\left(T\tau^{(333)} \right)^t = -T\tau^{(333)} \qquad\qquad \left(T\tau^{(322)} \right)^t = -T\tau^{(322)} \tag{D.2}$$

Using the antisymmetry of these matrices, it is manifest that the following combinations, involving the fluxes, will vanish,

$$
\begin{aligned}
0 &= \zeta^t T g_a \tau^{(322)} \sigma^2 \sigma^a \zeta = \zeta^t T h_a \tau^{(333)} \sigma^2 \sigma^a \zeta \\
0 &= \zeta^\dagger T g_a \tau^{(322)} \sigma^2 \sigma^a \zeta^* = \zeta^\dagger T h_a \tau^{(333)} \sigma^2 \sigma^a \zeta^* \\
0 &= \zeta^\dagger T g_a \tau^{(322)} (3\delta^{ab} + \sigma^{ab}) \zeta^* = \zeta^\dagger T h_a \tau^{(333)} (3\delta^{ab} + \sigma^{ab}) \zeta^*
\end{aligned}
\tag{D.3}
$$

When T commutes with $\tau^{(111)}$, these identities already follow from the chirality constraint (D.1). Thus, new identities will be obtained only for T anti-commuting with $\tau^{(111)}$, namely

$$T \in \mathcal{T} = \left\{ \tau^{(310)}, \ \tau^{(301)}, \ \tau^{(201)}, \ \tau^{(210)} \right\} \tag{D.4}$$

Using these combinations in the dilatino BPS equation, as well as in σ^3 times the dilatino BPS equation, we obtain two constraints,

$$\partial_a B \, \zeta^\dagger T \sigma^a \zeta = \partial_a B \, \zeta^\dagger T \sigma^3 \sigma^a \zeta = 0 \qquad\qquad T \in \mathcal{T} \tag{D.5}$$

Using the fact that $\sigma^3 \sigma^a = i\epsilon^{ab}\sigma^b$, with $\epsilon^{12} = 1$, we see that $\zeta^\dagger T \sigma^a \zeta$ dotted into $\partial_a B$ as well as into $\epsilon^{ab}\partial_b B$ vanishes. Whenever $\partial_a B \neq 0$, this implies (6.1).

D.1.1 The gravitino equations algebraic in ζ

We now construct another set of bilinear constraints. We multiply to the left the (m), (i_1) and (i_2) algebraic gravitino equations given in (5.10) by $\zeta^\dagger T \sigma^p$, for $p = 0, 3$, and use the fact that the g_a, h_a terms cancel and obtain, after some minimal simplifications,

$$
\begin{aligned}
(m) \qquad & 0 = -i\zeta^\dagger T \sigma^p \tau^{(211)} \zeta + \partial_a f_4 \zeta^\dagger T \sigma^p \sigma^a \zeta + f_4 f_a \zeta^\dagger T \sigma^p \tau^{(100)} \sigma^a \zeta \\
(i_1) \qquad & 0 = \zeta^\dagger T \sigma^p \tau^{(021)} \zeta + \partial_a f_1 \zeta^\dagger T \sigma^p \sigma^a \zeta - f_1 f_a \zeta^\dagger T \sigma^p \tau^{(100)} \sigma^a \zeta \\
(i_2) \qquad & 0 = \zeta^\dagger T \sigma^p \tau^{(002)} \zeta + \partial_a f_2 \zeta^\dagger T \sigma^p \sigma^a \zeta - f_2 f_a \zeta^\dagger T \sigma^p \tau^{(100)} \sigma^a \zeta
\end{aligned}
\tag{D.6}
$$

The terms multiplying $\partial_a f_1, \partial_a f_2, \partial_a f_4$ cancel via (6.1). Since for all $\tau \in \mathcal{T}$, $T\tau^{(100)}$ anti-commutes with $\tau^{(111)}$, it follows from (D.1) that the last term of each line vanishes, leaving three sets of new equations, for $p = 0, 3$,

$$\zeta^\dagger T \sigma^p \tau^{(211)} \zeta = \zeta^\dagger T \sigma^p \tau^{(021)} \zeta = \zeta^\dagger T \sigma^p \tau^{(002)} \zeta = 0 \tag{D.7}$$

or more explicitly, the constraints (6.2).

D.2 Solution to the T- and U-constraints

We begin by solving the U-constraints. Since the BPS equations are linear in ζ and ζ^*, the space of all ζ forms a vector space, in which linear combinations with real coefficients of solutions produce again solutions. Since $U\sigma^p$, for $p = 0, 3$ is Hermitian, it is immediate, from (6.2), that any two solutions ζ and ζ' must satisfy

$$(\zeta')^\dagger U\sigma^p \zeta = 0 \qquad\qquad p = 0, 3 \qquad\qquad (\text{D}.8)$$

Since the constraints hold for the insertions of both σ^0 and σ^3, they hold separately on the chiral components $\zeta_\pm$ of ζ, which obey $\sigma^3 \zeta_\pm = \pm\zeta_\pm$. The set of U matrices is invariant under multiplication by $\tau^{(011)}$. Thus, the U-constraints may be recast in the following form,

$$(\zeta'_\pm)^\dagger U_0 \left(1 \mp \tau^{(111)}\right) \left(1 + s\tau^{(011)}\right) \zeta_\pm = 0 \qquad\qquad (\text{D}.9)$$

for $s = \pm 1$ independently of $\pm$, and $U_0 \in \{\tau^{(001)}, \ \tau^{(220)}, \ \tau^{(212)}\}$. For any fixed ζ', these are linear projector equations for $\zeta_\pm$. The projector reduces the 2-dimensional space of ζ_+ to a 1-dimensional one.

D.2.1 Solving the two-dimensional reduced problem

The reduced problem (D.9) is effectively 2-dimensional, and thus takes the form

$$(\xi')^\dagger M\xi = 0, \qquad\qquad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \qquad\qquad \xi' = \begin{pmatrix} \xi'_1 \\ \xi'_2 \end{pmatrix} \qquad (\text{D}.10)$$

where $\xi_1, \xi_2, \xi'_1, \xi'_2 \in \mathbf{C}$, and M is a 2×2 Hermitian traceless matrix. Diagonalizing M by a unitary matrix u, we have $\xi = u\xi_0$, $\xi' = u\xi'_0$ and $M = \lambda u\tau^3 u^\dagger$ for some real constant λ. The condition $\xi_0^\dagger \tau^3 \xi_0 = 0$ has a one-dimensional vector space of solutions, whose basis spinor may be chosen to obey $\tau^1 \xi_0 = +\xi_0$. Other choices for the basis spinor are related to the above by a unitary transformation (and a scaling). For example, one could have chosen instead the basis spinor to be $\tilde{\xi}_0 = \tau^2 \xi_0$, so that $\tau^3 \tilde{\xi}_0 = -\tilde{\xi}_0$.

The results may be summarized as follows. The general solution to $\xi^\dagger M\xi = 0$ is completely characterized by a linear projector condition $P\xi = +\xi$ (or equivalently $P\xi = -\xi$), where P is Hermitian, and satisfies $P^2 = I$, and $\{M, P\} = 0$.

D.2.2 Solution of the U constraints

In view of the preceeding analysis, it suffices to find a Hermitian involution that anti-commutes with all the U matrices. There are 2 such matrices, $\tau^{(032)}$ and $\tau^{(123)}$, which, under multiplication by $\tau^{(111)}$, are equivalent to one another. Thus, the U-constraints are solved by requiring that ζ satisfy

$$\tau^{(032)} \zeta_\pm = \nu_\pm \zeta_\pm \qquad\qquad (\text{D}.11)$$

Here, ν_+ can be either $+1$ or -1 but not both. The same holds independently for ν_-.

D.2.3 Solution of the T constraints

Assuming that ζ satisfies the chirality condition $\sigma^3 \tau^{(111)} = -\zeta$, as well as the conditions (D.11), we now enforce the constraint $\zeta^\dagger T \sigma^a \zeta = 0$, for $a = 1, 2$, and $T \in \mathcal{T}$. Choosing instead the basis $a = +, -$, and using hermiticity of T, we get equivalently,

$$\zeta_\pm^\dagger \tau^{(310)} \left(1 \mp \tau^{(111)} \right) \left(1 + s\tau^{(011)} \right) \zeta_\mp = 0 \tag{D.12}$$

for $s = \pm 1$ independently of $\pm$. This problem is again 2-dimensional and we may take the representation $\tau^{(310)} \to \tau^3$ and $\tau^{(032)} \to \nu_+ \tau^1$. As a result, we have ξ_+ proportional to ξ_0, forcing also ξ_- to be proportional to ξ_0. Thus, ξ_- must be proportional to ξ_+, which gives the general solution for the T-constraint, (6.3), with $\nu_+ = \nu_-$.

E. Metric factors expressed solely in terms of h_1 and h_2

In analyzing the regularity conditions of the local solutions, it will be useful to have alternative expressions for the metric factors. To help simplify the presentation, we use the following definitions,

$$\begin{aligned} W &\equiv \partial_w h_1 \partial_{\bar{w}} h_2 + \partial_{\bar{w}} h_1 \partial_w h_2 \\ N_1 &\equiv 2 h_1 h_2 |\partial_w h_1|^2 - h_1^2 W \\ N_2 &\equiv 2 h_1 h_2 |\partial_w h_2|^2 - h_2^2 W \end{aligned} \tag{E.1}$$

The metric factors are derived from (6.26), first in terms of α and β, and then, using (7.9), in terms of λ, κ, ϕ and ρ. Finally, one eliminates ϕ using (9.28), ρ using (9.30), and κ and λ using (9.27). In the process, one must choose signs for various square roots. For example, to obtain ρ^2 from the expression for ρ^8 in (9.30), the first sign choice is dictated by the sign of W, while the second sign choice is determined by the sign of the product $h_1 h_2$. Without loss of generality, we may choose $N_2 > 0$ and $h_1 h_2 > 0$, by reversing the signs respectively of h_1 and h_2, if necessary. A sign choice for W must still be made, but this sign choice yields different formula for the AdS_4 metric factor f_4. For the case $W > 0$, the metric factors are given by

$$\begin{aligned} \rho^2 &= e^{-\phi} \frac{\sqrt{N_2 W}}{h_1 h_2} \\ f_1^2 &= 2 e^{-\phi} h_2^2 \sqrt{\frac{W}{N_2}} \\ f_2^2 &= 2 e^{+\phi} h_1^2 \sqrt{\frac{W}{N_1}} \\ f_4^2 &= 2 e^{3\phi} h_1^2 h_2^2 \frac{|\partial_w h_1 \partial_{\bar{w}} h_2 - \partial_{\bar{w}} h_1 \partial_w h_2|^2}{N_2 \sqrt{W N_2}} \end{aligned} \tag{E.2}$$

For the case $W < 0$, the metric factors are given by

$$\rho^2 = e^{-\phi} \frac{\sqrt{N_2 |W|}}{h_1 h_2}$$

$$f_1^2 = 2e^{+\phi} h_1^2 \sqrt{\frac{|W|}{N_1}}$$

$$f_2^2 = 2e^{-\phi} h_2^2 \sqrt{\frac{|W|}{N_2}}$$

$$f_4^2 = 2e^{-\phi} \sqrt{\frac{N_2}{|W|}} \tag{E.3}$$

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